COORDINATE GEOMETRY
(Polar Coordinates Approach)

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Chapter 1

Polar coordinate system

This chapter is introductory in nature. In Section 1.1, polar coordinates in the Euclidean plane are introduced. In Section 1.2, relations between Cartesian and polar coordinates are explained. Distance between two points is obtained in Section 1.3. Section 1.4 contains a formula for area of a triangle. In Example 4 of Section 1.5 area of a quadrilateral is obtained.

1.1 Polar coordinates

The position of a point in a two-dimensional Euclidean plane may be determined by means of a **polar coordinate system**. In brief, this is the system of coordinates in which a point is represented by its distance from a fixed point and the angle that the line from this point to the fixed point makes with a fixed line in anti-clockwise direction. We introduce a polar coordinate system as follows.

We fix a point O in the plane and call it the **pole**. A half-line OX starting from the pole is called the **polar axis** (or the **initial line**). A point P ($\neq O$) in the plane is uniquely determined by two numbers (see Figure 1.1): the number r which is the numerical distance OP of P from the pole O, and the number θ, which is the angle formed by the line segment OP with the polar axis OX in anti-clock wise direction. The numbers r and θ are called **polar coordinates** of the point P. The first coordinate r is called the **radius vector** and the second coordinate θ is called the **polar**

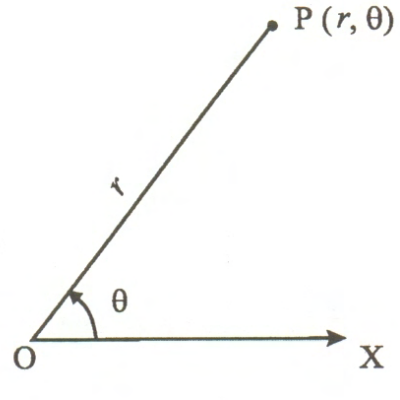


Figure 1.1: Polar Coordinates

angle or **vectorial angle** or **amplitude** of the point. The positive value of θ is reckoned counterclock-wise. For the pole, $r = 0$ and θ is arbitrary.

Now, let us allow the values of θ beyond $0 \leq \theta < 2\pi$, and the negative values of r. Let P be the point (r, θ) and Q be the point on the line PO produced such that $OP = OQ$ (see Figure 1.2). Then the polar coordinates of Q are $(r, \theta + \pi)$, which we may agree to write as $(-r, \theta)$. Thus the polar coordinates of P can be written as $(-r, \theta + \pi)$. In fact, the point P can be represented by an infinite number of polar coordinates as $(r, \theta \pm 2n\pi)$ and $(-r, \theta \pm (2n+1)\,\pi)$, $n = 0, 1, 2, \ldots$.

The consideration of points, with polar coordinates (r, θ) allowing r to be negative, might simplify the discussion of some special cases. However, we adopt in general the convention $r \geq 0$ and $0 \leq \theta < 2\pi$.

The equation to the pole is $r = 0$. The equation of the polar axis is $\theta = 0$. The equation of a straight line passing through the pole will be of the form $\theta =$ constant, since allowing negative values of the radius vector every point on the line will have the same vectorial angle.

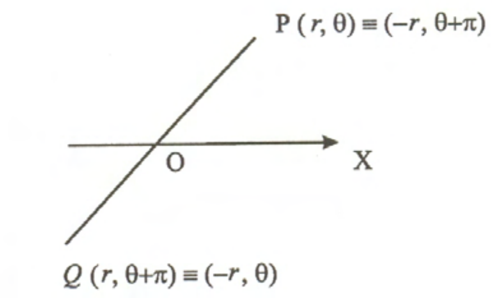


Figure 1.2: Different representations of polar coordinates of a point

1.2 Relation between Cartesian and polar coordinates

Consider a system of rectangular Cartesian coordinates and a system of polar coordinates. Let the origin of the rectangular Cartesian coordinate system coincide with the pole of the polar coordinate system. Furthermore, assume that the positive direction of the x-axis coincides with the polar axis, and that the positive direction of the y-axis coincides with the ray $\theta = \pi/2$. Then the equations of transformation from polar coordinates (r, θ) to Cartesian coordinates (x, y) of the same point are given by (see Figure 1.3)

$$x = r\cos\theta\,, \qquad y = r\sin\theta\,. \tag{1}$$

Conversely, the equations of transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ) of the same point may be written as

$$r = \sqrt{x^2 + y^2}\,, \qquad \theta = \arctan\left(\frac{y}{x}\right)\,. \tag{2}$$

Moreover,

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}\,, \qquad \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}\,. \tag{3}$$

Taking the radius vector to be nonnegative, r is determined uniquely from the relation (2). In determining the value of θ from

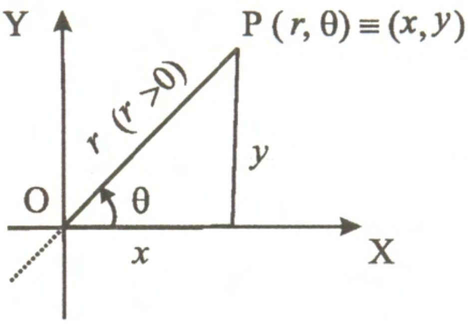


Figure 1.3: Cartesian and Polar coordinates

the relation (2), we get two values of θ, one in the first quadrant and the other in the third quadrant or one in the second quadrant and the other in the fourth quadrant according as y/x is positive or negative. Since the quadrant is fixed by the coordinates (x, y), therefore the correct value of θ may be fixed.

1.3 Distance between two points

Let $P_1\,(r_1, \theta_1)$ and $P_2\,(r_2, \theta_2)$ be two points. We join OP_1, OP_2 and P_1P_2, where O is the pole. Then $\angle P_1OP_2 = \theta_2 - \theta_1$.

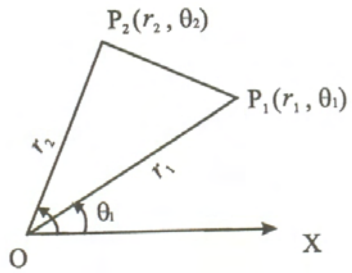


Figure 1.4: Distance between two points

Using cosine rule in the triangle OP_1P_2 we get (see Figure 1.4)

$$\cos\angle P_1OP_2 = \frac{{OP_1}^2 + {OP_2}^2 - {P_1P_2}^2}{2OP_1 \cdot OP_2} = \frac{{r_1}^2 + {r_2}^2 - {P_1P_2}^2}{2r_1r_2},$$

which gives

$$P_1P_2 = \left({r_1}^2 + {r_2}^2 - 2r_1r_2\cos(\theta_1 - \theta_2)\right)^{1/2}.$$

1.4 Area of a triangle

Let O be the pole and $P_i\,(r_i, \theta_i)$, $i = 1, 2, 3$ be the vertices the triangle $P_1P_2P_3$ taken in anti-clockwise order.

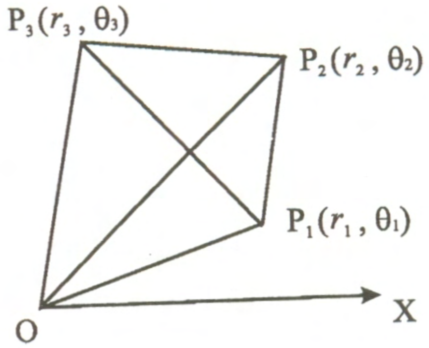


Figure 1.5: Area of a triangle

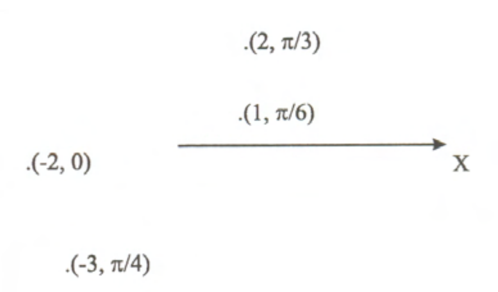
Then, we have (see Figure 1.5)

$$\begin{aligned}
& \text{area of triangle } P_1P_2P_3 \\
= \;& \text{area of triangle } OP_1P_2 + \text{area of triangle } OP_2P_3 \\
& - \text{area of triangle } OP_1P_3 \\
= \;& \frac{1}{2}r_1r_2\sin(\theta_2 - \theta_1) + \frac{1}{2}r_2r_3\sin(\theta_3 - \theta_2) - \frac{1}{2}r_1r_3\sin(\theta_3 - \theta_1) \\
= \;& \frac{1}{2}\left(r_1r_2\sin(\theta_2 - \theta_1) + r_2r_3\sin(\theta_3 - \theta_2) + r_3r_1\sin(\theta_1 - \theta_3)\right) \\
= \;& \frac{1}{2}\left(r_1r_2\sin(\theta_2 - \theta_1) + r_2r_3\sin(\theta_3 - \theta_2) + r_3r_1\sin(\theta_1 - \theta_3)\right).
\end{aligned}$$

1.5 Solved Examples

Example 1. Represent the points whose polar coordinates are
(a) $(2, \pi/3)$, (b) $(-2, 0)$, (c) $(1, \pi/6)$, (d) $(-3, \pi/4)$.

Solution.



Example 2(a). Consider the parabola $y^2 = 4ax$. Putting $x = r\cos\theta$ and $y = r\sin\theta$, the polar equation of the parabola becomes $r^2\sin^2\theta = 4ar\cos\theta$ or $r = 4a\cos\theta/\sin^2\theta$. □
2(b). Consider the equation $l/r = 1+e\cos\theta$ or $r^2-(l-er\cos\theta)^2 = 0$ and using $r^2 = x^2+y^2$ and $x = r\cos\theta$ we get $x^2+y^2-(l-ex)^2 = 0$. □

Example 3. Show that the points $(0, 0)$, $(a, \pi/6)$ and $(a, \pi/2)$ are vertices of an equilateral triangle.

Solution. Let the given points be $P\,(0, 0)$, $Q\,(a, \pi/6)$ and $R\,(a, \pi/2)$. We know that the distance between points (r_1, θ_1) and (r_2, θ_2) is

$$\left({r_1}^2 + {r_2}^2 - 2r_1r_2\cos(\theta_1 - \theta_2)\right)^{1/2}.$$

The statement $PQ = a = PR$ is obvious. We also have

$$QR = \left(a^2 + a^2 - 2\,(a)\,(a)\cos(\pi/2 - \pi/6)\right)^{1/2} = a.$$

Hence the points P, Q and R are vertices of an equilateral triangle.
□

Example 4. Let O be the pole and $P_i\,(r_i, \theta_i)$, $i = 1, 2, 3, 4$ be the vertices the quadrilateral $P_1P_2P_3P_4$ taken in anti-clockwise order.

Then, we have

$$
\begin{aligned}
& \text{area of quadrilateral } P_1P_2P_3P_4 \\
= & \text{ area of triangle } P_1P_2P_3 + \text{area of triangle } P_3P_4P_1 \\
= & \frac{1}{2}\left(r_1r_2\sin\left(\theta_2-\theta_1\right)+r_2r_3\sin\left(\theta_3-\theta_2\right)+r_3r_1\sin\left(\theta_1-\theta_3\right)\right) \\
& +\frac{1}{2}\left(r_3r_4\sin\left(\theta_4-\theta_3\right)+r_4r_1\sin\left(\theta_1-\theta_4\right)+r_1r_3\sin\left(\theta_3-\theta_1\right)\right) \\
= & \frac{1}{2}\left(r_1r_2\sin\left(\theta_2-\theta_1\right)+r_2r_3\sin\left(\theta_3-\theta_2\right)\right. \\
& \quad \left.+\,r_3r_4\sin\left(\theta_4-\theta_3\right)+r_4r_1\sin\left(\theta_1-\theta_4\right)\right).
\end{aligned}
$$

Example 5. Let P, Q, R be points on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with centre C. If CP, CQ, CR are inclined at angle $2\pi/3$ to each other, then show that

$$\frac{1}{CP^2}+\frac{1}{CQ^2}+\frac{1}{CR^2}=\frac{3}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right).$$

Solution. Putting $x = r\cos\theta$ and $y = r\sin\theta$ in the equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ we get the polar equation of the ellipse

$$\frac{r^2\cos^2\theta}{a^2}+\frac{r^2\sin^2\theta}{b^2}=1$$

or

$$\frac{1}{r^2}=\frac{1}{b^2}+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos^2\theta. \tag{1}$$

Let vectorial angles of P, Q, R be θ, $(\theta+2\pi/3)$, $(\theta+4\pi/3)$ respectively. Then in view of equation (1) we get

$$\frac{1}{CP^2}+\frac{1}{CQ^2}+\frac{1}{CR^2}$$

$$=\frac{3}{b^2}+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\cos^2\theta+\cos^2\left(\frac{\pi}{3}-\theta\right)+\cos^2\left(\frac{\pi}{3}+\theta\right)\right) \tag{2}$$

Now, we have

$$
\begin{aligned}
& \cos^2\theta + \cos^2\left(\frac{\pi}{3} - \theta\right) + \cos^2\left(\frac{\pi}{3} + \theta\right) \\
= \; & \cos^2\theta + \left(\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right)^2 + \left(\cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta\right)^2 \\
= \; & \cos^2\theta + \left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)^2 + \left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right)^2 \\
= \; & \cos^2\theta + \left(\frac{1}{4}\cos^2\theta + \frac{3}{4}\sin^2\theta + \frac{\sqrt{3}}{2}\cos\theta\sin\theta\right) \\
& + \left(\frac{1}{4}\cos^2\theta + \frac{3}{4}\sin^2\theta - \frac{\sqrt{3}}{2}\cos\theta\sin\theta\right) \\
= \; & \frac{3}{2}\left(\cos^2\theta + \sin^2\theta\right) = \frac{3}{2}.
\end{aligned}
$$

Putting this value in equation (2) we get

$$\frac{1}{CP^2} + \frac{1}{CQ^2} + \frac{1}{CR^2} = \frac{3}{b^2} + \frac{3}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{3}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

1.6 Problem Set 1

1. Represent the points whose polar coordinates are

   (**a**) $(2, -\pi/2)$, (**b**) $\left(\sqrt{3}, 2\pi/3\right)$,
   (**c**) $(1/2, -\pi/6)$, (**d**) $(-1, \pi/4)$.

2. Locate the points in the plane whose polar coordinates satisfy the following conditions:

   (**a**) $r = 2$, (**b**) $1 < r < 2$, (**c**) $\theta = \pi/6$,
   (**d**) $\pi/3 < \theta < \pi/4$, (**e**) $2 \leq r < 4$.

3. Find the polar coordinates of the point symmetric to the point $(1, \pi/3)$ relative to

   (**a**) the pole,
   (**b**) the polar axis.

4. Find the polar coordinates of the points:

   (**a**) $(-1, 1)$, (**b**) $(-1, -1)$,
   (**c**) $\left(-1, \sqrt{3}\right)$, (**d**) $(3, -4)$.

5. Find the Cartesian coordinates of the points:

   (**a**) $\left(2, \frac{\pi}{4}\right)$, (**b**) $\left(3, \frac{\pi}{2}\right)$.

6. Transform the following Cartesian equations into the polar form

   (**a**) $x^2 + y^2 - 5x = 0$,
   (**b**) $x^2 = 4ay$,
   (**c**) $\dfrac{x^2}{a^2} \pm \dfrac{y^2}{b^2} = 1$.

7. Change the following polar equations into Cartesian form:

   (**a**) $r^2 = a\cos 2\theta$,
   (**b**) $r = 2a\cos\theta$,
   (**c**) $\dfrac{l}{r} = 1 + e\cos(\theta - \alpha)$.

8. Find the distance between points $P(a, \theta)$ and $Q(-a, \pi + \theta)$.

9. Find the distance between points $P(3, 50°)$ and $Q(4, 110°)$.

10. Show that the points $(a, \pi/3)$ and $(b, \pi/3)$ are collinear with the pole.

11. Show that the points $(1, \pi/7)$, $(4, \pi/7)$ and $(7, -6\pi/7)$ are collinear.

12. Find the area of the triangle with vertices

    (**i**) $(3, \pi/6)$, $(2, \pi/2)$ and $(1, 5\pi/6)$,
    (**ii**) (a, θ), $(2a, \theta + \pi/3)$ and $(3a, \theta + 2\pi/3)$,
    (**iii**) $(-a, \pi/6)$, $(a, \pi/2)$ and $(-2a, -2\pi/3)$.

13. Show that the points $(0, 0)$, (a, α) and $(a, \alpha + \pi/3)$ are vertices of an equilateral triangle.

14. If r_1 and r_2 are two mutually perpendicular radius vectors of the ellipse
$$r^2 = \frac{b^2}{1 - e^2 \cos^2 \theta},$$
then show that
$$\frac{1}{{r_1}^2} + \frac{1}{{r_2}^2} = \frac{1}{a^2} + \frac{1}{b^2},$$
where $b^2 = a^2\,(1 - e^2)$.

□ □ □

Chapter 2

Straight lines

The straight lines are studied using polar coordinates. In Section 2.1, general equation of a straight line is written. Section 2.2 deals with parallel and perpendicular lines, while in Section 2.3, equation of a line in normal form is obtained. In section 2.4, equation of a line joining two points is obtained. In Section 2.5, distance of a point from a line is determined. In Example 2 of Section 2.6, point of intersection of two lines is demonstrated, while in Example 3, three lines equally inclined to each other are discussed.

2.1 General equation of a straight line

We know that the general equation of a straight line in the rectangular Cartesian coordinate system can be written as

$$ax + by = l\,, \qquad (l \geq 0)\ .$$

Taking the pole at the origin and the positive x-axis as the polar axis, the above equation, in polar coordinates, becomes

$$\frac{l}{r} = a\cos\theta + b\sin\theta\,. \tag{1}$$

Thus the equation (1) is the general equation of a straight line in polar coordinates. Keeping l fixed and varying a and b only, we can get different straight lines. If $l = 0$, the straight line passes through the pole and its equation is

$$a\cos\theta + b\sin\theta = 0 \quad \text{or} \quad \tan\theta = \text{constant} \quad \text{or} \quad \theta = \text{constant}. \quad (2)$$

The polar equation of a straight line may also be taken as

$$\frac{l}{r} = a\cos(\theta - \alpha) + b\cos(\theta - \psi)\ , \qquad (3)$$

where α and ψ are fixed.

2.2 Parallel and perpendicular lines

Since $ax + by = l$ and $ax + by = l'$ represent parallel lines, therefore the lines

$$\frac{l}{r} = a\cos\theta + b\sin\theta \qquad \text{and} \qquad \frac{l'}{r} = a\cos\theta + b\sin\theta$$

are parallel lines. Since the equations

$$ax + by = l \quad \text{and} \quad bx - ay = l'$$

express two mutually perpendicular lines, therefore in the polar coordinates these mutually perpendicular lines are

$$\frac{l}{r} = a\cos\theta + b\sin\theta \qquad \text{and} \qquad \frac{l'}{r} = a\cos\left(\theta + \frac{\pi}{2}\right) + b\sin\left(\theta + \frac{\pi}{2}\right)\ .$$

It is easy to verify that the lines

$$\frac{l}{r} = a\cos\theta + b\sin\theta \qquad \text{and} \qquad \frac{l'}{r} = a'\cos\theta + b'\sin\theta$$

are mutually perpendicular if $aa' + bb' = 0$.

2.3 Straight lines in normal form

Now, we find the equation of a straight line, in terms of the perpendicular $ON = p\ (> 0)$ from the pole on it and the angle α which the perpendicular ON makes with the polar axis OX. Let $P(r, \theta)$ be any point on the straight line.

We consider the right angled triangle ONP. Then we have $ON/OP = \cos\angle PON$, that is, (see Figure 2.1)

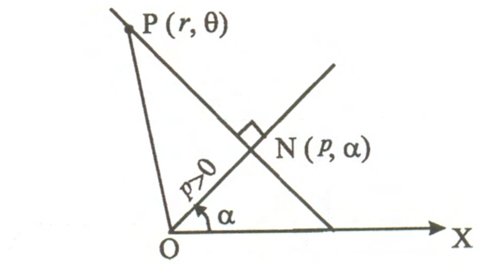


Figure 2.1: Straight line: $p/r = \cos(\theta - \alpha)$

$$\frac{p}{r} = \cos|(\theta - \alpha)| = \cos(\theta - \alpha)\ . \tag{1}$$

In particular, if $\alpha = 0$ then we get (see Figure 2.2)

$$\frac{p}{r} = \cos|\theta| = \cos\theta. \tag{2}$$

If $\alpha = \pi/2$, that is, the line is parallel to the polar axis and is above the polar axis, then the line becomes (see Figure 2.3)

$$\frac{p}{r} = \cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta\,. \tag{3}$$

If the parallel line is below the polar axis, then $\alpha = 3\pi/2$ (or $-\pi/2$), and in this case the equation of the line is

$$\frac{p}{r} = -\sin\theta\,. \tag{4}$$

A line perpendicular to the line $p/r = \cos(\theta - \alpha)$ is given by

$$\frac{p'}{r} = \cos\left(\theta - \alpha + \frac{\pi}{2}\right)\ . \tag{5}$$

2.4 A straight line joining two points

Let $P_i\,(r_i, \theta_i)$, $i = 1, 2$ be two points. Let $P\,(r, \theta)$ be any point on the line P_1P_2. Since P_1, P_2 and P are colinear, therefore area of the triangle P_1P_2P is zero, that is

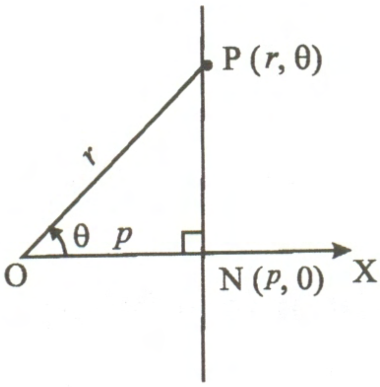


Figure 2.2: Straight line: $p/r = \cos\theta$

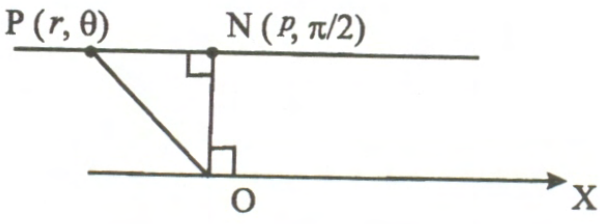


Figure 2.3: Straight line: $p/r = \sin\theta$

$$r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r \sin(\theta - \theta_2) + r r_1 \sin(\theta_1 - \theta) = 0. \qquad (1)$$

Hence equation (1) is the required equation of the straight line joining two points $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$.

2.5 Distance of a point from a line

Let $P_1(r_1, \theta_1)$ be a point and

$$\frac{l}{r} = a\cos\theta + b\sin\theta$$

be a line. In Cartesian coordinates, the point is $P_1(r_1\cos\theta_1, r_1\sin\theta_1)$ and the line is $ax + by = l$. Therefore, the required distance is

$$\left|\frac{ar_1\cos\theta_1 + br_1\sin\theta_1 - l}{\sqrt{a^2+b^2}}\right|.$$

2.6 Solved Examples

Example 1. Find the length and direction of the perpendicular from the pole to the line

$$\frac{5}{r} = 4\cos\theta - 3\sin\theta.$$

Solution. This line can be put in form

$$\frac{1}{r} = \frac{4}{5}\cos\theta - \frac{3}{5}\sin\theta = \cos(\theta - \alpha), \qquad \alpha = \arctan\left(\frac{-3}{4}\right).$$

Hence, length of perpendicular from the pole is 1 and its direction is $\arctan(-3/4)$. □

Example 2. Find the polar coordinates of the point of intersection of the two different straight lines

$$\frac{l}{r} = \cos\theta + \cos(\theta - 2\alpha) \quad \text{and} \quad \frac{l}{r} = \cos\theta + \cos(\theta - 2\beta).$$

Solution. Let the two lines meet in a point $P(\rho, \phi)$. Then, we get

$$\frac{l}{\rho} = \cos\phi + \cos(\phi - 2\alpha) = \cos\phi + \cos(\phi - 2\beta).$$

From the above equation we get

$$\cos(\phi - 2\alpha) = \cos(\phi - 2\beta) \quad \text{or} \quad (\phi - 2\alpha) = \pm(\phi - 2\beta).$$

Taking positive sign we get $\alpha = \beta$, which makes the two lines identical, so taking negative sign we get $\phi = \alpha+\beta$. Putting $\theta = \phi = \alpha+\beta$, and $r = \rho$ in the equation of one of the two given lines we get

$$\frac{l}{\rho} = \cos(\alpha + \beta) + \cos(\beta - \alpha) = 2\cos\alpha\cos\beta,$$

which gives $\rho = (l/2)\sec\alpha\sec\beta$. Hence the point of intersection of the two lines is

$$\left(\frac{l}{2}\sec\alpha\sec\beta, \alpha + \beta\right). \ \square$$

Example 3. Show that the equation

$$y^3 - 3x^2y + m\left(x^3 - 3xy^2\right) = 0$$

represents three straight lines equally inclined to one another.

Solution. We change the given equation to polar form:

$$r^3\sin^3\theta - 3r^3\cos^2\theta\sin\theta + mr^3\left(\cos^3\theta - 3\cos\theta\sin^2\theta\right) = 0$$

or

$$3\sin\theta\cos^2\theta - \sin^3\theta = m\left(\cos^3\theta - 3\cos\theta\sin^2\theta\right)$$

or

$$3\tan\theta - \tan^3\theta = m\left(1 - 3\tan^2\theta\right)$$

or

$$m = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \qquad \text{or} \qquad \tan 3\theta = m = \tan\alpha, \quad \text{say.}$$

Then

$$3\theta = n\pi + \alpha \qquad \text{or} \qquad \theta = (n\pi + \alpha)/3.$$

Putting $n = 1, 2, 3$, we get

$$\theta_1 = \frac{\pi}{3} + \frac{\alpha}{3}, \quad \theta_2 = \frac{2\pi}{3} + \frac{\alpha}{3}, \quad \theta_3 = \frac{3\pi}{3} + \frac{\alpha}{3}.$$

The angles between these lines being equal to $\frac{\pi}{3}$, the given lines are equally inclined to one another. □

2.7 Problem Set 2

1. Find the length and direction of the perpendicular from the pole to the line $30/r = 4\cos\theta + 3\sin\theta$.

2. Find the vectorial angle of the point of intersection of two straight lines

   $$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \quad \text{and} \quad \frac{l}{r} = e\cos\theta + \cos(\theta - \beta).$$

   When will these lines be perpendicular?

3. Show that the lines $1/r = 2\sin\theta + 4\cos\theta$ and $5/r = 3\cos\theta - 6\sin\theta$ are mutually perpendicular.

4. Show that the equation of a straight line joining two points (r_1, θ_1) and (r_2, θ_2) is

   $$\begin{vmatrix} 1/r & \cos\theta & \sin\theta \\ 1/r_1 & \cos\theta_1 & \sin\theta_1 \\ 1/r_2 & \cos\theta_2 & \sin\theta_2 \end{vmatrix} = 0.$$

   Hence or otherwise show that the three points (r_i, θ_i), $i = 1, 2, 3$ are collinear if

   $$0 = \begin{vmatrix} 1/r_1 & \cos\theta_1 & \sin\theta_1 \\ 1/r_2 & \cos\theta_2 & \sin\theta_2 \\ 1/r_3 & \cos\theta_3 & \sin\theta_3 \end{vmatrix} = \sum \frac{1}{r_1}\sin(\theta_2 - \theta_3).$$

5. Show that the equation

$$\frac{l}{r} = f(\theta) \equiv a\cos(\theta+\alpha) + b\cos(\theta+\beta)$$

represents the equation of a straight line, and that any straight line perpendicular to it is $l'/r = f(\theta + \pi/2)$.

6. Find the equation of bisectors of the lines $\theta = \alpha$ and $\theta = \beta$.

7. Find the polar coordinates of the point of intersection of the two different straight lines

$$\frac{l}{r} = \cos\theta + \cos(\theta-\alpha) \quad \text{and} \quad \frac{l}{r} = \cos\theta + \cos(\theta-\beta).$$

8. Find the polar coordinates of the point of intersection of the two different straight lines

$$\frac{l}{r} = \cos\theta - \cos(\theta-\alpha) \quad \text{and} \quad \frac{l}{r} = \cos\theta - \cos(\theta-\beta).$$

9. Prove that the straight line passing through the point of intersection of the line $1/r = a\cos\theta + b\sin\theta$ and $1/r = a'\cos\theta + b'\sin\theta$ is the straight line

$$\frac{1+\lambda}{r} = (a+\lambda a')\cos\theta + (b+\lambda b')\sin\theta.$$

Hence or otherwise, obtain the equation of the line passing through the pole and the point of intersection of the line $1/r = a\cos\theta + b\sin\theta$ and $1/r = a'\cos\theta + b'\sin\theta$.

10. Find the distance of a point (r_1, θ_1) from a line $r\cos(\theta-\alpha) = p$.

11. Show that the polar coordinates of the foot of perpendicular from the pole on the line joining two points $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ is

$$\left(\frac{r_1 r_2 \sin(\theta_1-\theta_2)}{\sqrt{{r_1}^2+{r_2}^2-2r_1 r_2\cos(\theta_1-\theta_2)}}, \arctan\frac{r_2\cos\theta_2 - r_1\cos\theta_1}{r_1\sin\theta_1 - r_2\sin\theta_2}\right).$$

12. If the vectorial angle of a point P on the line joining two points $P_1(r_1,\theta_1)$ and $P_2\,(r_2,\theta_2)$ is $(\theta_1+\theta_2)\,/2$, then show that the length of the radius vector of P is
$$\frac{2r_1r_2}{r_1+r_2}\cos\left(\frac{\theta_1-\theta_2}{2}\right).$$

13. Show that the three lines
$$r\cos\left(\theta-\alpha\right)=a, r\cos\left(\theta-\beta\right)=b\text{and}r\cos\left(\theta-\gamma\right)=c$$
are concurrent if and only if
$$a\sin\left(\beta-\gamma\right)+b\sin\left(\gamma-\alpha\right)+c\sin\left(\alpha-\beta\right)=0.$$

14. Find the angle between the three straight lines given by the equation
$$x^3+3x^2y-3xy^2-y^3=0.$$

15. Show that the equation
$$\sin 3\alpha\left(y^3-3x^2y\right)+\cos 3\alpha\left(x^3-3xy^2\right)+3a\left(x^2+y^2\right)-4a^3=0$$
represents three straight lines, which form an equilateral triangle.

□ □ □

Chapter 3

Circles

Circles are studied in this chapter. In Section 3.1, general equation of a circle with given centre and given radius is obtained. Some particular cases are also demonstrated. In Section 3.2, we obtain equation of a chord of a circle. In Sections 3.3 and 3.4 equations of tangent and normal at a point of a circle are obtained respectively. In Example 2 of Section 3.5, we obtain an equation a circle circumscribing a triangle. Condition of tangency of a line to circles are discussed in Examples 4 and 5. In Example 6, it is proved that locus of the feet of perpendiculars from the pole on tangents to a circle is a cardioid.

3.1 Equation of a circle

Simplest equation of a circle is when its centre is at the pole. $r = a$ is the equation in this case, where a is the radius of this circle.

Next, we take the general case, when the centre is $C\,(c, \alpha)$. In fact, we obtain the polar equation of a circle whose centre is $C\,(c, \alpha)$ and radius is a. Let $P\,(r, \theta)$ be any point on the circle. Then $OC = c$, $OP = r$, $CP = a$, $\angle COX = \alpha$ and $\angle POX = \theta$ (see Figure 3.1).

Using the cosine formula in the triangle OCP, we have

$$CP^2 = OP^2 + OC^2 - 2OP \cdot OC \cos \angle POC\,,$$

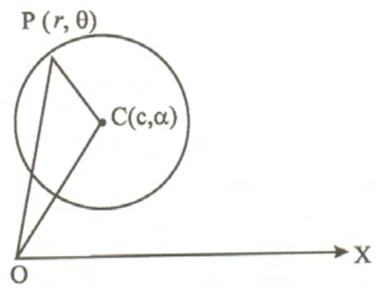


Figure 3.1: Circle: $r^2 - 2cr\cos(\theta - \alpha) + c^2 - a^2 = 0$

that is,

$$a^2 = r^2 + c^2 - 2rc\cos(\theta - \alpha)\ .$$

Therefore, the equation of the circle is

$$r^2 - 2cr\cos(\theta - \alpha) + c^2 - a^2 = 0\,. \tag{1}$$

Particular cases

Case 1. If the circle touches the initial line, then we have $a = |c\sin\alpha|$ and the equation of the circle reduces to (see Figure 3.2)

$$r^2 - 2cr\cos(\theta - \alpha) + c^2\cos^2\alpha = 0\,. \tag{2}$$

Case 2. If the centre lies on the polar axis, then we have $\alpha = 0$ and the equation of the circle reduces to (see Figure 3.3)

$$r^2 - 2cr\cos\theta + c^2 - a^2 = 0\,. \tag{3}$$

Case 3. If the pole lies on the circle, then we have $c = a$ and the equation of the circle reduces to (see Figure 3.4)

$$r = 2a\cos(\theta - \alpha)\,. \tag{4}$$

Case 4. If the pole lies on the circle and the polar axis is the diameter through that point, then it is the Case 3 with $\alpha = 0$ and the equation of the circle reduces to (see Figure 3.5)

$$r = 2a\cos\theta. \tag{5}$$

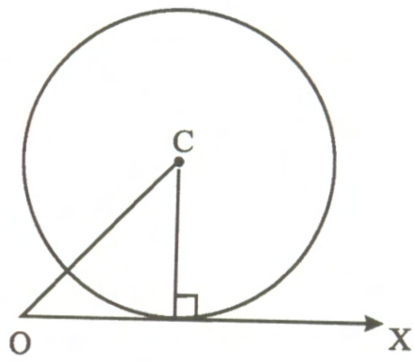


Figure 3.2: Circle: $r^2 - 2cr\cos(\theta - \alpha) + c^2\cos^2\alpha = 0$.

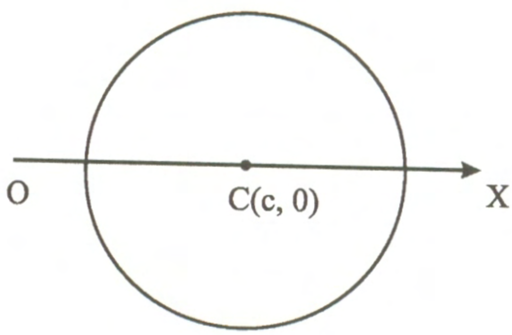


Figure 3.3: Circle: $r^2 - 2cr\cos\theta + c^2 - a^2 = 0$

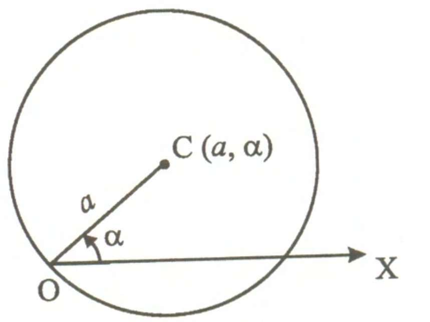


Figure 3.4: Circle: $r = 2a\cos(\theta - \alpha)$

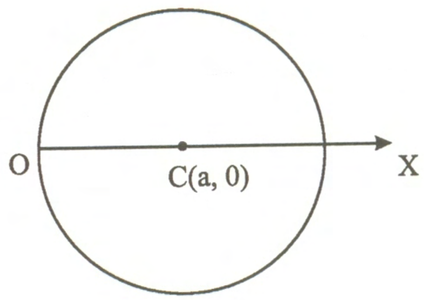


Figure 3.5: Circle: $r = 2a\cos\theta$

Case 5. If the pole lies on the circle and the diameter of the circle through the pole is perpendicular to the polar axis, then in the Case 3 we have $\alpha = \pi/2$ and the equation of the circle is (see Figure 3.6)

$$r = 2a \sin \theta \, .$$

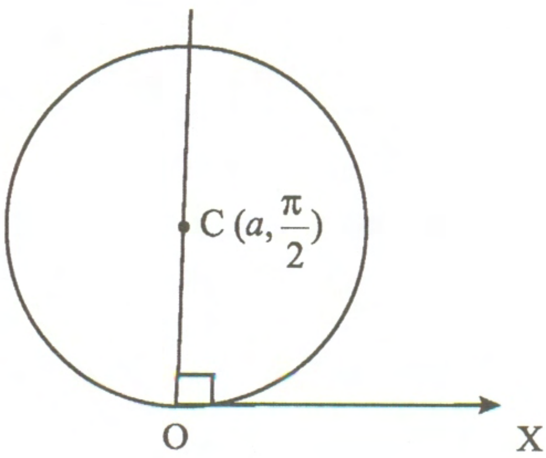


Figure 3.6: Circle: $r = 2a \sin \theta$

Case 6. If the pole coincides with the centre of the circle, then the equation of the circle is $r = a$.

3.2 Chord of the circle

Now, we obtain polar equation of the chord joining two points $P\,(r_1, \theta_1)$ and $Q\,(r_2, \theta_2)$ on the circle $r = 2a \cos \theta$.

We draw the perpendicular OM to the chord PQ (See Figure 3.7). If polar coordinates of M are (p, α) then the equation of the chord PQ is

$$p = r \cos (\theta - \alpha) \; .$$

Now, we intend to find out p and α. First, we note that

$$\angle MOQ = \angle PQA = \angle POA = \theta_1 \, .$$

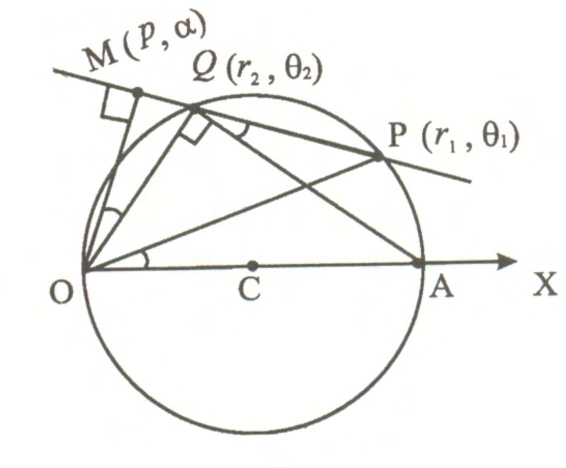


Figure 3.7: Chord of circle

Thus we get

$$\alpha = \angle MOA = \angle MOQ + \angle QOA = \theta_1 + \theta_2 \,,$$

and

$$\begin{aligned} p &= OM = OQ \cos \angle MOQ \\ &= OA \cos \angle QOA \cos \angle MOQ \\ &= 2a \cos \theta_2 \cos \theta_1 \,. \end{aligned}$$

Thus polar coordinates of the foot of perpendicular M from the pole on the chord joining $P\,(\theta_1)$ and $Q\,(\theta_2)$ on the circle $r = 2a\cos\theta$ are

$$M\,(2a\cos\theta_1\cos\theta_2, \theta_1{+}\theta_2)\,.$$

Hence, the equation of the chord PQ is

$$r\cos(\theta - \theta_1 - \theta_2) = 2a\cos\theta_1\cos\theta_2\,. \tag{1}$$

Similarly, the equation of the chord joining the two points θ_1 and θ_2 on the circle $r = 2a\cos(\theta - \psi)$ can be obtained from the equation (1) by changing θ, θ_1 and θ_2 to $\theta - \psi$, $\theta_1 - \psi$ and $\theta_2 - \psi$ respectively:

$$r\cos(\theta - \theta_1 - \theta_2 + \psi) = 2a\cos(\theta_1 - \psi)\cos(\theta_2 - \psi)\ . \tag{2}$$

3.3 Tangent at a point of the circle

Here we obtain the polar equation of the tangent at a point $T(\alpha)$ on the circle $r = 2a\cos\theta$.

The tangent at $T(\alpha)$ is the limiting case of the chord joining the points $P(\theta_1)$ and $Q(\theta_2)$ as $\theta_1 \to \alpha$ and $\theta_2 \to \alpha$. Therefore, taking $\theta_1 = \alpha = \theta_2$ in the equation

$$r\cos(\theta - \theta_1 - \theta_2) = 2a\cos\theta_1\cos\theta_2$$

of the chord PQ we get the equation of the tangent at the point $T(\alpha)$ as

$$r\cos(\theta - 2\alpha) = 2a\cos^2\alpha\,.$$

Remark. Equation of the tangent at the point $T(\alpha)$ on the circle $r = 2a\cos(\theta - \psi)$ can be obtained from the above equation by changing θ to $\theta - \psi$ and α to $\alpha - \psi$. Thus the required equation of the tangent is

$$r\cos(\theta + \psi - 2\alpha) = 2a\cos^2(\alpha - \psi)\,.$$

3.4 Normal at a point of the circle

Since the normal at $T(\alpha)$ is perpendicular to the tangent at $T(\alpha)$, therefore its equation can be taken as

$$r\cos\left(\theta - 2\alpha + \frac{\pi}{2}\right) = p'\,,$$

where p' is to be determined. Since this normal passes through the centre $C(a, 0)$ of the circle, therefore

$$p' = a\cos\left(-2\alpha + \frac{\pi}{2}\right) = a\sin 2\alpha\,.$$

Therefore, the polar equation of the normal at the point $T(\alpha)$ on the circle $r = 2a\cos\theta$ is

$$r\sin(2\alpha - \theta) = a\sin 2\alpha\,.$$

Remark. Equation of the normal at the point $T(\alpha)$ on the circle $r = 2a\cos(\theta - \psi)$ can be obtained from the above equation by

changing θ to $\theta - \psi$ and α to $\alpha - \psi$. Thus the required equation of the normal is

$$r \sin(2\alpha - \theta - \psi) = a \sin 2(\alpha - \psi) \ .$$

3.5 Solved Examples

Example 1. In general, the equation $r = a\cos\theta + b\sin\theta$ represents a circle through the pole. In fact, we can write the above equation as

$$r = d\cos(\theta - \alpha)$$

with

$$d = \left(a^2 + b^2\right)^{1/2} \quad \text{and} \quad \alpha = \arctan\left(\frac{b}{a}\right) ,$$

which represents a circle through the pole with the centre $\left(\frac{d}{2}, \alpha\right)$.
□

Example 2. Show that polar equation of the circle circumscribing the triangle with vertices $P\left(\frac{l}{2}\sec\alpha\sec\beta, \alpha+\beta\right)$,
$Q\left(\frac{l}{2}\sec\beta\sec\gamma, \beta+\gamma\right)$ and $R\left(\frac{l}{2}\sec\gamma\sec\alpha, \gamma+\alpha\right)$ is

$$2r\cos\alpha\cos\beta\cos\gamma = l\cos(\theta - \alpha - \beta - \gamma) .$$

Solution. The given equation can be written as

$$r = \left(\frac{l}{2}\sec\alpha\sec\beta\sec\gamma\right)\cos(\theta - (\alpha + \beta + \gamma)) ,$$

which is a circle. It can be verified that the points
$P\left(\frac{l}{2}\sec\alpha\sec\beta, \alpha+\beta\right)$, $Q\left(\frac{l}{2}\sec\beta\sec\gamma, \beta+\gamma\right)$
and $R\left(\frac{l}{2}\sec\gamma\sec\alpha, \gamma+\alpha\right)$ satisfy this equation. □

Example 3. Let O be a fixed point and P be any point on a given circle. OP is joined and a point Q is taken on OP such that

$OP \cdot OQ = k$. Show that the locus of Q is a circle. Further, show that the locus of Q is a straight line if the point O is taken on the circle.

Solution. We take the fixed point O as the pole and the line joining O with the centre C of the circle as the initial line. Let $(c, 0)$ be coordinates of C and a be the radius of the circle $(c > a)$. Then the equation of the circle is

$$r^2 - 2cr\cos\theta + c^2 - a^2 = 0. \qquad (1)$$

Let $P\,(r_1, \theta_1)$ be a point on the circle (1), so that

$$r_1{}^2 - 2cr_1\cos\theta_1 + c^2 - a^2 = 0. \qquad (2)$$

Let $Q\,(\rho, \theta_1)$ be a point on the line OP such that

$$k = OP \cdot OQ = r_1\rho \quad \text{or} \quad r_1 = \frac{k}{\rho}.$$

Substituting this value of r_1 in (2), we get

$$\left(\frac{k}{\rho}\right)^2 - 2c\frac{k}{\rho}\cos\theta_1 + c^2 - a^2 = 0$$

or

$$\rho^2\left(c^2 - a^2\right) - 2ck\rho\cos\theta_1 + k^2 = 0.$$

Therefore the locus of the point Q is

$$r^2\left(c^2 - a^2\right) - 2ckr\cos\theta + k^2 = 0,$$

which is a circle. If the pole O lies on the circle (1) then $c = a$ and the above equation reduces to $2cr\cos\theta = k$,which is a straight line. □

Example 4. Find the condition so that the straight line

$$\frac{l}{r} = a\cos\theta + b\sin\theta$$

may touch the circle $r = 2c\cos\theta$.

Solution. Centre of the circle is $(c, 0)$. Perpendicular from $(c, 0)$ to the straight line is

$$\frac{|l - ac|}{\sqrt{a^2 + b^2}},$$

which must be c. Thus

$$(l - ac)^2 = c^2\left(a^2 + b^2\right) \qquad \text{or} \qquad b^2c^2 + 2alc = l^2. \qquad \square$$

Example 5. Find the condition so that the line $r\cos(\theta - \alpha) = p$ may touch the circle

$$r^2 - 2rc\cos\theta + c^2 - a^2 = 0.$$

Solution. The given circle is

$$r^2 - 2rc\cos\theta + c^2 - a^2 = 0, \tag{1}$$

whose centre is $(c, 0)$ and radius is a. The given line is

$$r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p. \tag{2}$$

Perpendicular on (2) from $(c, 0)$ is

$$|c\cos\alpha - p|.$$

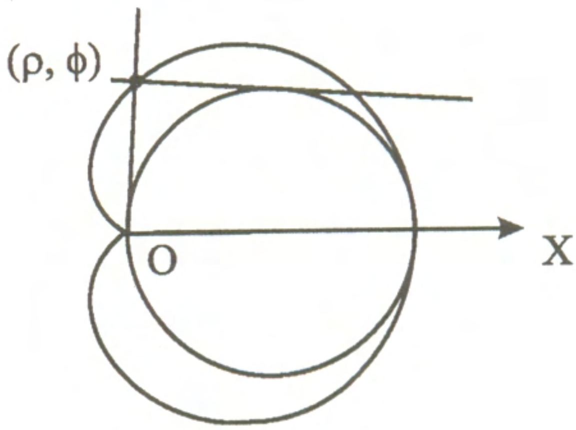
Hence for (2) to touch (1) the condition is

$$|c\cos\alpha - p| = a. \qquad \square$$

Example 6. Locus of the feet of perpendiculars from the pole on tangents to the circle $r = 2a\cos\theta$ is the cardioid $r = a(1 + \cos\theta)$.

Solution. Let

$$r\cos(\theta - 2\alpha) = 2a\cos^2\alpha \tag{1}$$



be a tangent at a point α on the circle $r = 2a\cos\theta$. Let (ρ, ϕ) be the foot of perpendicular drawn from the pole to the tangent. Then

$$\rho\cos(\phi - 2\alpha) = 2a\cos^2\alpha. \tag{2}$$

Since the point (ρ, ϕ) also lies on the line

$$r\cos(\theta - 2\alpha + \pi/2) = 0 \tag{3}$$

perpendicular to (1) and passing through the pole, therefore

$$\rho\cos(\phi - 2\alpha + \pi/2) = 0. \tag{4}$$

Since, in general $\rho \neq 0$, therefore from (4) it follows that $2\alpha = \phi$. Putting $2\alpha = \phi$ in the equation (2) we get

$$\rho = 2a\cos^2(\phi/2) = a(1 + \cos\phi) \ ,$$

which shows that the locus of the point (ρ, ϕ) is the cardioid

$$r = a(1 + \cos\theta) \,. \qquad \square$$

3.6 Problem Set 3

1. Express the following circles in the polar form:

   (a) $x^2 + y^2 - ax = 0$,

   (b) $x^2 + y^2 - by = 0$.

2. Show that $r^2 + 2r(a\cos\theta + b\sin\theta) + c = 0$ represents a circle.

3. Find the centre and radii of the circles

   (a) $r^2 - r(4\cos\theta + 3\sin\theta) - 2 = 0$,

   (b) $r = 7\left(\cos\theta + \sqrt{3}\sin\theta\right)$,

   (c) $r = 3(\cos\theta + \sin\theta)$,

   (d) $r = 6\left(\sqrt{3}\cos\theta + \sin\theta\right)$.

4. Obtain the polar equation of the circle when initial line is a tangent to the circle.

5. A circle passes through the point (r_1, θ_1) and touches the initial line at a distance c from the pole. Show that its polar equation is

$$\frac{r^2 - 2cr\cos\theta + c^2}{r\sin\theta} = \frac{{r_1}^2 - 2cr_1\cos\theta_1 + c^2}{r_1\sin\theta_1}.$$

6. Find the polar equation of the circle passing through the points $(a, 0°)$, $(b, 0°)$ and touching the line $\theta = \alpha$.

7. Obtain the polar equation of the circle drawn on the line segment joining points (a, α) and (b, β) as diameter.

8. Show that the length of the diameter of the circle passing through the pole and the points $P_1\,(r_1, \theta_1)$ and $P_2\,(r_2, \theta_2)$ is

$$\frac{\left({r_1}^2 + {r_2}^2 - 2r_1r_2\cos\left(\theta_1 - \theta_2\right)\right)^{1/2}}{\sin\left(\theta_1 - \theta_2\right)}.$$

9. Show that polar equation of the circle circumscribing the polygon of n sides with vertices

$$P_1\left(l\sec\alpha_2\sec\alpha_3\cdots\sec\alpha_n, \alpha_2 + \alpha_3 + \cdots + \alpha_n\right),$$

$$P_2\left(l\sec\alpha_1\sec\alpha_3\cdots\sec\alpha_n, \alpha_1 + \alpha_3 + \cdots + \alpha_n\right),$$

$$P_3\left(l\sec\alpha_1\sec\alpha_2\sec\alpha_4\cdots\sec\alpha_n, \alpha_1+\alpha_2+\alpha_4+\cdots+\alpha_n\right), \ldots,$$

$$P_n\left(l\sec\alpha_1\sec\alpha_2\cdots\sec\alpha_{n-1}, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}\right) \text{ is}$$

$$r\cos\alpha_1\cos\alpha_2\cdots\cos\alpha_n = l\cos\left(\theta - \alpha_1 - \alpha_2 - \cdots - \alpha_n\right).$$

10. Show that the feet of the perpendiculars from the pole on the sides of the triangle formed by the points (r_i, θ_i), $i = 1, 2, 3$, on the circle $r = 2a\cos\theta$ lie on the line

$$r\cos\left(\theta - \theta_1 - \theta_2 - \theta_3\right) = 2a\cos\theta_1\cos\theta_2\cos\theta_3.$$

This line is the pedal line of the pole with respect to the triangle.

11. A triangle of the given species has one vertex at the pole and a second vertex moving on the circle

$$r^2 - 2dr\cos\theta + d^2 - a^2 = 0.$$

Show that the locus of the third vertex is the circle

$$k^2r^2 - 2dkr\cos(\theta - \alpha) + d^2 - a^2 = 0$$

for suitable constants k and α.

12. Show that the equations

$$r^2 - kr\cos(\theta - \alpha) + kd = 0$$

represent a coaxial system of circles for different values of k. Show that the radical axis of the system is $r\cos(\theta - \alpha) = d$ and the limiting points are $(0, \alpha)$ and $(2d, \alpha)$.

13. P is any point on the circle S and P' is the inverse of P with respect to a fixed circle S_0 with centre at the pole. Prove that as P moves on S, P' moves on a circle S' which may reduce to a straight line in a special case, but which in any case belongs to the coaxial system of circles determined by S and S_0.

14. Show that the triangle formed by the pole and the points of intersection of the line $r\cos\theta = 3$ and the circle $r = 4\cos\theta$ is equilateral.

15. Find the equation of the tangent at a point with vectorial angle $\pi/4$ on the circle $r = 2a\cos\theta$.

16. Show that the tangent at (r_1, θ_1) to the circle

$$r^2 - 2cr\cos(\theta - \alpha) + c^2 - a^2 = 0$$

is

$$rr_1\cos(\theta - \theta_1) - rc\cos(\theta - \alpha) = {r_1}^2 - r_1c\cos(\theta_1 - \alpha).$$

17. Show that the condition for the straight line

$$\frac{1}{r} = a\cos\theta + b\sin\theta$$

to touch the circle $r = 2c\cos\theta$ is $b^2c^2 + 2ac = 1$.

18. Find the condition for the straight line

$$\frac{1}{r} = (1/a)\cos\theta + (1/b)\sin\theta$$

to touch the circle $r = 2c\cos\theta$.

19. Find the equation for determining the radius vectors of the points of intersection of the circle $r = 2a\cos\theta$ and the line $r\cos(\theta - \alpha) = p$. For what value of p the straight line is a tangent to the circle?

20. Show that the line $r\cos\theta = c + a$ touches the circle

$$r^2 - 2cr\cos\theta + c^2 - a^2 = 0.$$

Find the point of contact.

21. Find the angle of intersection of the circles $r = a\cos(\theta - \alpha)$ and $r = b\sin(\theta - \alpha)$.

22. Show that the circles $r = a\cos(\theta - \alpha)$ and $r = b\sin(\theta - \beta)$ intersect at an angle $(\alpha - \beta)$.

□ □ □

Chapter 4

Conics

A conic is the locus of a point which moves so that the ratio of its distance from a fixed point to its distance from a fixed line is a (positive finite) constant. The ratio is the **eccentricity of the conic**, the fixed point the **focus**, and the fixed line the **directrix**. The eccentricity is usually denoted by e. The conic is an ellipse, a parabola or a hyperbola according as $0 < e < 1$, $e = 1$ or $1 < e < +\infty$ respectively. The polar equation of a conic is simplest if its focus (or one of its foci) is at the pole. In this case the polar equation of the conic reduces to very simple form which is widely used in the suitable problems.

In this chapter, conics are studied using polar coordinates. In Section 4.1, we determine equation of a conic in polar coordinates. Several particular cases are considered. Tracing of conics are discussed in Section 4.2. Directrix of a conic is obtained in Section 4.3. Example 2 of Section 4.4 states that the semi-latus rectum of a conic is the harmonic mean between the segments of a focal chord, while in Example 4, some properties of a pair of mutually perpendicular focal chords are given. In Section 4.6, we find equation of a chord of a conic. Section 4.9, deals with the tangent of a conic. Condition of tangency is discussed in Section 4.10. Equations of chord of contact and pair of tangents are obtained in Sections 4.11 and 4.12 respectively. In Section 4.13, asymptotes of a conic are discussed. Normal at a point of the conic is obtained in Section 4.16. The chapter concludes with an objective test in Section 4.19.

4.1 Equation of a conic

We find the polar equation of a conic taking pole at focus and polar axis making an angle ψ with the axis of the conic in anti-clockwise. Let e and l denote eccentricity and semi-latus rectum of the conic respectively.

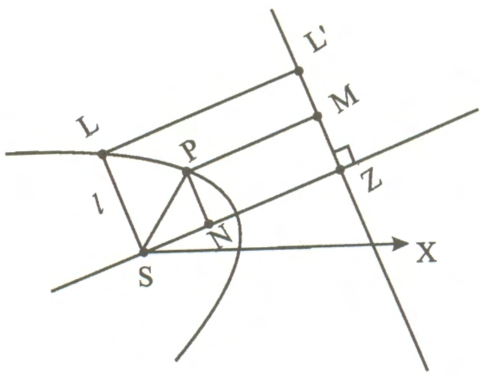


Figure 4.1: Conic: $l/r = 1 + e\cos(\theta - \psi)$

Let S be the focus coinciding with the pole and let the axis SZ of the conic make an angle ψ with the polar axis Sx. Let $P(r, \theta)$ be a point on the conic. We draw perpendiculars PM and PN from P on the directrix ZM and the axis SZ respectively. Let $SL\,(= l)$ be the semi-latus rectum of the conic and $LL'\,(= SZ)$ be the perpendicular from L to the directrix ZM (See Figure 4.1).

By definition of the conic, we have

$$r = SP = ePM = e(SZ - SN) = eLL' - eSN = l - er\cos(\theta - \psi)\,.$$

Hence the polar equation of the conic is

$$\frac{l}{r} = 1 + e\cos(\theta - \psi)\,. \tag{1}$$

If the distance SZ between the focus S and the fixed line ZM is d then the above equation of the conic can be written as

$$\frac{ed}{r} = 1 + e\cos(\theta - \psi)\,. \tag{2}$$

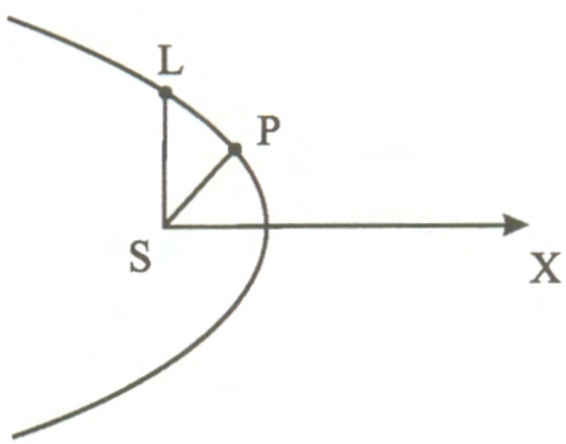


Figure 4.2: Conic: $l/r = 1 + e\cos\theta$

Particular cases
Case 1. If the axis SZ of the conic coincides with the polar axis (that is, $\psi = 0$), then its equation becomes (See Figure 4.2)

$$\frac{l}{r} = 1 + e\cos\theta\,. \tag{3}$$

In general, for a conic this equation is used.

Case 2. If the positive direction of the axis is opposite to the polar axis, (that is, $\psi = \pi$), then the equation of the conic is (See Figure 4.3)

$$\frac{l}{r} = 1 - e\cos\theta\,. \tag{4}$$

4.2 Tracing of conics

Since $\cos(-\theta) = \cos\theta$, therefore the equation of the conic $l/r = 1 + \cos\theta$ remains unaltered when we replace θ by $-\theta$. Hence the curve

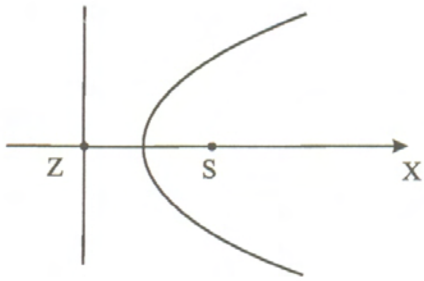


Figure 4.3: Conic: $l/r = 1 - e\cos\theta$

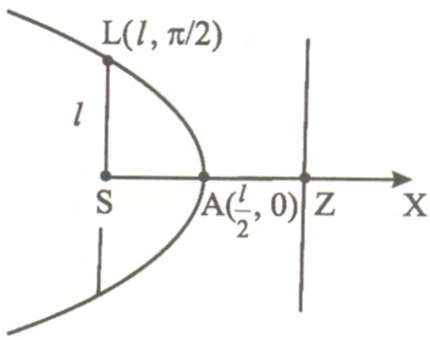


Figure 4.4: Parabola: $l/r = 1 + \cos\theta$

is symmetrical about the polar axis. According to the behaviour of the eccentricity e we consider the following three cases:

Case 1 – Parabola. When $e = 1$, the equation of the conic reduces to the parabola

$$\frac{l}{r} = 1 + \cos\theta \qquad \text{or} \qquad r = \frac{l}{1+\cos\theta}.$$

As θ increases from 0 to $\pi/2$, $\cos\theta$ decreases from 1 to 0 and hence r increases from $l/2$ to l. Further, as θ increases from $\pi/2$ to π, $\cos\theta$ decreases from 0 to -1 and hence r increases from l to infinity. The rest of the curve follows from the symmetry about the polar axis (see Figure 4.4).

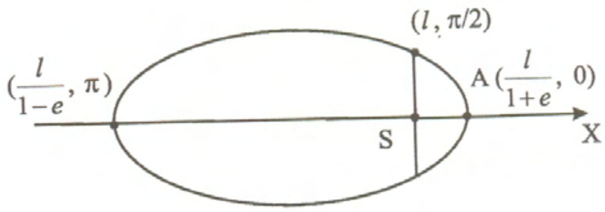


Figure 4.5: Ellipse: $l/r = 1 + e\cos\theta$

Case 2 – Ellipse. When $0 < e < 1$, the conic is an ellipse

$$r = \frac{l}{1+e\cos\theta}.$$

Since $0 < e < 1$, therefore $1 + e\cos\theta$ is never zero and hence r is always finite. Thus the ellipse is a closed curve. For $\theta = 0$, we get $r = l/(1+e)$. As θ increases from 0 to $\pi/2$, $1+e\cos\theta$ decreases from $1 + e$ to 1, and therefore r increases from $l/(1 + e)$ to l. Further, as θ increases from $\pi/2$ to π, $1 + e\cos\theta$ decreases from 1 to $1 - e$ and hence r increases from l to $l/(1 - e)$. Rest of the curve can be traced by symmetry about the initial line (see Figure 4.5). We note that as $e \to 0$, the ellipse approaches the circle $r = l$.

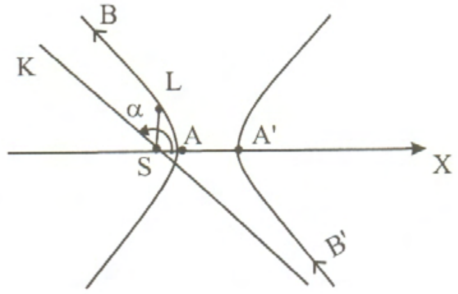


Figure 4.6: Hyperbola : $l/r = 1 + e\cos\theta$

Case 3 – Hyperbola. When $e > 1$, the conic is the hyperbola

$$r = \frac{l}{1 + e\cos\theta}.$$

For $\theta = 0$, we get $r = l/(1+e)$, that is, the hyperbola meets the axis at the point A such that $SA = l/(1+e)$. As θ increases from 0 to $\pi/2$, $1 + e\cos\theta$ decreases from $1 + e$ to 1, and therefore r increases from $l/(1+e)$ to l. Let α be the obtuse angle satisfying the equation $\alpha = \cos^{-1}(-1/e) = \angle ASK$, say. Thus for $\theta = \alpha$, $1 + e\cos\theta$ vanishes and r approaches infinity. Hence as θ increases from $\pi/2$ to α, r increases from l to infinity. This part is indicated by the portion ALB in the Figure 4.6. Further, as θ increases from α to π, $1 + e\cos\theta$ decreases from 0 to $1 - e$ and hence r increases from minus infinity to $-l/(e-1)$. The numerical value of r is to be measured in the opposite direction to that of the radius vector. This part of the hyperbola is indicated by the portion $B'A'$ in the Figure 4.6. Rest of the hyperbola follows from symmetry about the initial line (see Figure 4.6).

4.3 Directrices of a conic

We show that the two directrices of the conic $l/r = 1 + e\cos\theta$ are $l/r = e\cos\theta$ and $l/r = ((e^2-1)/(e^2+1))e\cos\theta$(see Figure 4.7).

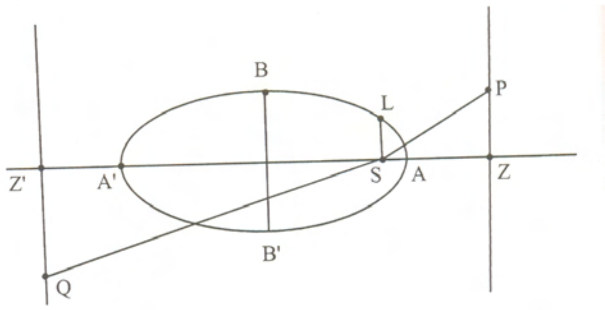


Figure 4.7: Directrices of a conic

If $P(r,\theta)$ is a point on the directrix corresponding to the focus S then its equation is

$$r\cos\theta = SZ\,.$$

Since $SZ = SL/e = l/e$, therefore the above equation becomes

$$\frac{l}{r} = e\cos\theta\,. \tag{1}$$

Assuming the conic as an ellipse, now we obtain the equation of the other directrix. However, the result is true in the case of a hyperbola also. Let $Q(r,\theta)$ be a point on the other directrix. Since SZ' is the perpendicular distance of this directrix from the focus and SZ' makes an angle π from the polar axis, therefore the equation of this directrix is

$$r\cos(\theta-\pi) = SZ' \quad \text{or} \quad r\cos\theta = -SZ'.$$

But

$$l = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2)$$

and therefore

$$SZ' = ZZ' - SZ = \frac{2a}{e} - \frac{l}{e} = \frac{2l}{e(1-e^2)} - \frac{l}{e} = -\left(\frac{e^2+1}{e^2-1}\right)\frac{l}{e}\,.$$

Putting this value of SZ' in the equation $r\cos\theta = -SZ'$ of the directrix we get

$$\frac{l}{r} = \left(\frac{e^2-1}{e^2+1}\right) e\cos\theta\,. \tag{2}$$

4.4 Solved Examples

Example 1. A point P moves such that the sum of its distances from two fixed points S and S' is a constant $2a$. Then the locus of the point P is an ellipse.

Solution. Let S be the pole and SS' be the initial line. Let $P(r,\theta)$ be any point such that $SP + S'P = 2a$. Then $SS' < 2a$; and let polar coordinates of S' be $(2ae, 0)$, where $e < 1$. Using cosine rule in the triangle SPS' we get

$$S'P^2 = SP^2 + SS'^2 - 2SP \cdot SS' \cos\theta$$

or

$$(2a - r)^2 = r^2 + (2ae)^2 - 2r(2ae)\cos\theta,$$

which implies that $a(1 - e^2) = r(1 - e\cos\theta)$. □

Example 2. The semi-latus rectum of a conic is the harmonic mean between the segments of a focal chord.

Solution. Let PSQ be a focal chord of a conic $l/r = 1 + e\cos\theta$. Let the vectorial angle of the point P be α. Then the vectorial angle of the point Q will be $(\alpha + \pi)$. Since P and Q both lie on the given conic, therefore

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1 + e\cos\alpha}{l} + \frac{1 + e\cos(\alpha + \pi)}{l} = \frac{2}{l}. \qquad \square$$

Example 3. If a circle of diameter a passing through the focus of a conic whose latus rectum is $2l$ meets the conic in four points whose distances from the focus are r_1, r_2, r_3 and r_4, then prove that

$$\textbf{(i)} \quad r_1 r_2 r_3 r_4 = \frac{a^2 l^2}{e^2}, \qquad \textbf{(ii)} \quad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l}.$$

Solution. Let the conic be $l/r = 1 + e\cos\theta$. The equation of the circle passing through the pole may be taken as

$$r = a\cos(\theta - \alpha) = a(\cos\theta\cos\alpha + \sin\theta\sin\alpha),$$

where a is the diameter of the circle and α is the angle which the diameter through the pole makes with the initial line. From the above circle, we get

$$(r - a\cos\theta\cos\alpha)^2 = a^2\sin^2\alpha(1-\cos^2\theta).$$

Hence substituting for $\cos\theta$ from the equation of the conic, we have

$$e^2r^4+2r^3ae\cos\alpha+r^2(a^2-2ael\cos\alpha-e^2a^2\sin^2\alpha)-2a^2lr+a^2l^2=0,$$

which gives the focal distances r_1, r_2, r_3, r_4 of the four points of intersection. Hence, we have

$$r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4 = -\frac{\text{coefficient of } r}{\text{coefficient of } r^4} = \frac{2a^2l}{e^2}\ , \quad (1)$$

$$r_1r_2r_3r_4 = \frac{\text{coefficient of } r^0}{\text{coefficient of } r^4} = \frac{a^2l^2}{e^2}. \quad (2)$$

The equation (2) gives the first part. Dividing the equation (1) by the equation (2) we get the second part as

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4} = \frac{2}{l}\,. \qquad \square$$

Example 4. If PSP' and QSQ' are two mutually perpendicular focal chords of a conic $l/r = 1+e\cos\theta$ then prove that

(a) $$\frac{1}{SP\cdot SP'}+\frac{1}{SQ\cdot SQ'} = \frac{2-e^2}{l^2},$$

(b) $$\frac{1}{PP'}+\frac{1}{QQ'} = \frac{2-e^2}{2l},$$

(c) $$\frac{(SP+SP'+SQ+SQ')(SP\cdot SP'\cdot SQ\cdot SQ')}{(SP+SP')(SQ+SQ')(SP\cdot SP'+SQ\cdot SQ')} = \frac{l}{2}.$$

Solution. Let the vectorial angle of P be α. Then

$$\frac{l}{SP} = 1+e\cos\alpha, \quad \frac{l}{SP'} = 1+e\cos(\alpha+\pi) = 1-e\cos\alpha,$$

$$\frac{l}{SQ} = 1 + e\cos\left(\alpha + \frac{\pi}{2}\right) = 1 - e\sin\alpha,$$
$$\frac{l}{SQ'} = 1 + e\cos\left(\alpha + \frac{\pi}{2} + \pi\right) = 1 + e\sin\alpha.$$

Hence

$$\frac{l^2}{SP \cdot SP'} + \frac{l^2}{SQ \cdot SQ'} = (1 - e^2\cos^2\alpha) + (1 - e^2\sin^2\alpha) = 2 - e^2$$

or

$$\frac{1}{SP \cdot SP'} + \frac{1}{SQ \cdot SQ'} = \frac{2 - e^2}{l^2}, \qquad (1)$$

which proves the first part. Next,

$$PP' = SP + SP' = \frac{l}{1 + e\cos\alpha} + \frac{l}{1 - e\cos\alpha} = \frac{2l}{1 - e^2\cos^2\alpha},$$
$$QQ' = SQ + SQ' = \frac{l}{1 - e\sin\alpha} + \frac{l}{1 + e\sin\alpha} = \frac{2l}{1 - e^2\sin^2\alpha},$$

which shows that

$$\frac{1}{PP'} + \frac{1}{QQ'} = \frac{1 - e^2\cos^2\alpha}{2l} + \frac{1 - e^2\sin^2\alpha}{2l} = \frac{2 - e^2}{2l},$$

or

$$\frac{1}{PP'} + \frac{1}{QQ'} = \frac{2 - e^2}{2l}, \qquad (2)$$

which is the second part. Dividing equation (2) by equation (1) side by side, we get the third part. □

Example 5. An ellipse and a parabola have a common focus S and intersect in two real points P and Q of which P is the vertex of the parabola. If e is the eccentricity of the ellipse and α the angle which SP makes with the major axis, then prove that

$$\frac{SQ}{SP} = 1 + \frac{4e^2\sin^2\alpha}{(1 - e\cos\alpha)^2}.$$

Solution. Let the parabola and the ellipse be

$$\frac{l}{r} = 1 + \cos\theta, \tag{1}$$

$$\frac{l'}{r} = 1 + e\cos(\theta - \alpha), \tag{2}$$

It is given that (1) and (2) intersect in the points P and Q where P is the vertex of the parabola. Therefore

$$SP = \frac{l}{2} = \frac{l'}{1 + e\cos\alpha}\,. \tag{3}$$

If β is the vectorial angle of Q then

$$\frac{l}{SQ} = 1 + \cos\beta \qquad \text{and} \qquad \frac{l'}{SQ} = 1 + e\cos(\beta - \alpha)$$

or

$$\frac{1}{SQ} = \frac{1 + \cos\beta}{l} = \frac{1 + e\cos(\beta - \alpha)}{l'}. \tag{4}$$

From (3) and (4) we get $\dfrac{1 + \cos\beta}{2} = \dfrac{1 + e\cos(\beta - \alpha)}{1 + e\cos\alpha}$, or

$$\begin{aligned}\cos^2\frac{\beta}{2}\,(1 + e\cos\alpha) &= 1 + e\left(\cos\alpha\left(2\cos^2\frac{\beta}{2} - 1\right)\right. \\ &\qquad + \left.\sin\alpha\left(2\sin\frac{\beta}{2}\cos\frac{\beta}{2}\right)\right).\end{aligned}$$

or

$$\sin^2\frac{\beta}{2} + e\cos\alpha\cos^2\frac{\beta}{2} - e\cos\alpha + 2e\sin\alpha\sin\frac{\beta}{2}\cos\frac{\beta}{2} = 0$$

or

$$\tan^2\frac{\beta}{2} + e\cos\alpha - e\cos\alpha\left(1 + \tan^2\frac{\beta}{2}\right) + 2e\sin\alpha\tan\frac{\beta}{2} = 0$$

or

$$(1 - e\cos\alpha)\tan^2\frac{\beta}{2} + 2e\sin\alpha\tan\frac{\beta}{2} = 0 \quad \text{or} \quad \tan\frac{\beta}{2} = \frac{-2e\sin\alpha}{1 - e\cos\alpha}\,.$$

Now from (4) we get

$$\frac{SQ}{l} = \frac{1}{1 + e\cos\beta} = \frac{1}{2}\sec^2\frac{\beta}{2} = \frac{1}{2}\left(1 + \tan^2\frac{\beta}{2}\right)$$

and from (3) we get $SP/l = 2$. Hence

$$\frac{SQ}{SP} = 1 + \tan^2 \frac{\beta}{2} = 1 + \frac{4e^2 \sin^2 \alpha}{(1 - e\cos\alpha)^2}\,. \qquad \square$$

4.5 Problem Set 4

1. Obtain the points on the conic $21/r = 3 - 8\cos\theta$ whose radius vector is 3.

2. Show that

   (a) the equations $\frac{l}{r} = 1 + e\cos\theta$ and $\frac{l}{r} = -1 + e\cos\theta$ represent the same conic.

   (b) the equations $\frac{l}{r} = 1 - e\cos\theta$ and $\frac{l}{r} = -1 - e\cos\theta$ represent the same conic.

3. Prove that the curve given by $\frac{1}{r} = a\cos\theta + b\sin\theta + c$ represents

   (a) a conic with origin as a focus, if a, b and c are nonzero,

   (b) a circle, if $a = b = 0 \neq c$,

   (c) a straight line, if $a \neq 0 \neq b$, and $c = 0$.

4. Identify the following curves:

(a) $r\sin\theta + a = 0,$ (b) $r = (a/2)\cos\theta,$

(c) $r^2\sin 2\theta - a^2 = 0,$ (d) $\dfrac{2a}{r} = 1 - \cos\theta,$

(e) $\dfrac{2}{r} = 1 + 2\cos^2(\theta/2),$ (f) $\dfrac{l}{r} = 3 - 2\cos\theta,$

(g) $\dfrac{2}{r} = -1/2 + (1/4)\cos\theta,$ (h) $r = 5\csc^2(\theta/2),$

(i) $\dfrac{l}{r} = 1 + \sin\theta,$ (j) $\dfrac{1}{r} = \cos^2(\theta/2),$

(k) $\dfrac{3}{r} = 2 + \sqrt{3}\cos\theta + \sin\theta,$ (l) $\dfrac{10}{r} = 3\cos\theta + 4\sin\theta + 5,$

(m) $\dfrac{2}{r} = 1 + \cos\theta + \sin\theta,$ (n) $\dfrac{12}{r} = 4 + \sqrt{3}\cos\theta + 3\sin\theta,$

(o) $\dfrac{1}{r} = 1 + \cos\theta + \sqrt{3}\sin\theta,$ (p) $\dfrac{18}{r} = 3 - 4\cos\theta,$

(q) $r\left(1 + \sqrt{2}\cos\theta\right) = 3,$ (r) $\dfrac{16}{r} = 4 + \sqrt{3}\cos\theta + 3\sin\theta.$

5. A straight line drawn through the common focus S of n conics meet them in the points $P_1, P_2, \ldots, P_n$. A point P on this straight line is taken such that

$$\frac{1}{SP} = \frac{1}{SP_1} + \frac{1}{SP_2} + \cdots + \frac{1}{SP_n}.$$

Prove that the locus of P is a conic whose focus is S and the reciprocal of whose latus rectum is equal to the sum of the reciprocals of the letera-recta of the given conics.

6. Let the circle $r + 2a\cos\theta = 0$ cut the conic

$$\frac{l}{r} = 1 + e\cos(\theta - \alpha)$$

in four points. If the algebraic sum of distances of these four points from the pole is equal to $2a$, then show that the eccentricity of the conic is $2\cos\alpha$.

7. Find the points of intersection of the parabolas $1/r = 1 + \cos\theta$ and $3/r = 1 - \cos\theta$.

8. Prove that the locus of the mid points of the focal chords of a conic section is a conic section of the same kind.

9. Find the length of the focal chord making an angle of $\pi/4$ with the major axis of an ellipse with latus rectum 10 and eccentricity $1/3$.

10. Let P be any point on an ellipse with major axis $2a$ and latus rectum $2l$. If PSQ and PHR are two focal chords through the foci S and H , then prove that

$$\frac{PS}{SQ}+\frac{PH}{HR}=\frac{2\,(2a-l)}{l},$$

which is independent of the position of P.

11. Prove that the perpendicular focal chords of a rectangular hyperbola are equal.

12. The eccentric angle of any point P on an ellipse is α, measured from the semi-major axis CA; S is the focus nearest to A and $\angle ASP=\theta$. Show that

$$\tan\frac{\theta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{\alpha}{2}.$$

(This relation is of importance in the theory of elliptic orbits in dynamics.)

4.6 Equation of a chord

Let $P(\alpha-\beta)$ and $Q(\alpha+\beta)$ be two points on the conic

$$\frac{l}{r}=1+e\cos\theta. \tag{1}$$

Let the equation of a line not passing through the pole be

$$\frac{l}{r}=a\cos\theta+b\sin\theta. \tag{2}$$

Since both P and Q lie on (1) and (2), therefore

$$\frac{l}{SP} = 1 + e\cos(\alpha - \beta) = a\cos(\alpha - \beta) + b\sin(\alpha - \beta) \tag{3}$$

and

$$\frac{l}{SQ} = 1 + e\cos(\alpha + \beta) = a\cos(\alpha + \beta) + b\sin(\alpha + \beta). \tag{4}$$

From (3) and (4), we get

$$\begin{aligned}(a - e)\cos(\alpha - \beta) + b\sin(\alpha - \beta) &= 1,\\ (a - e)\cos(\alpha + \beta) + b\sin(\alpha + \beta) &= 1.\end{aligned}$$

These equations will determine a and b. We have

$$\begin{aligned}&(a - e)\left(\cos(\alpha - \beta)\sin(\alpha + \beta) - \cos(\alpha + \beta)\sin(\alpha - \beta)\right)\\ &\qquad = \sin(\alpha + \beta) - \sin(\alpha - \beta)\end{aligned}$$

or

$$(a - e)\sin\left((\alpha + \beta) - (\alpha - \beta)\right) = 2\cos\alpha\sin\beta$$

or

$$(a - e) = \cos\alpha\sec\beta;$$

and

$$\begin{aligned}&b\left(\sin(\alpha + \beta)\cos(\alpha - \beta) - \sin(\alpha - \beta)\cos(\alpha + \beta)\right)\\ &\qquad = \cos(\alpha - \beta) - \cos(\alpha + \beta)\end{aligned}$$

or

$$b\sin 2\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

or

$$b = \sin\alpha\sec\beta.$$

Putting the values of a and b, the chord (2) becomes

$$\begin{aligned}\frac{l}{r} &= (e + \cos\alpha\sec\beta)\cos\theta + \sin\alpha\sec\beta\sin\theta\\ &= e\cos\theta + \sec\beta\left(\cos\alpha\cos\theta + \sin\alpha\sin\theta\right)\end{aligned}$$

or

$$\frac{l}{r} = \sec\beta\cos(\theta - \alpha) + e\cos\theta. \tag{5}$$

Remarks. If the vectorial angles of the points P and Q on the conic are θ_1 and θ_2 respectively, then the equation of the chord PQ is

$$\frac{l}{r} = \sec\frac{\theta_2-\theta_1}{2}\,\cos\left(\theta - \frac{\theta_1+\theta_2}{2}\right) + e\,\cos\theta\,. \tag{6}$$

If the given conic is $l/r = 1 + e\cos(\theta-\psi)$ then the equation of the chord joining two points $P(\alpha-\beta)$ and $P(\alpha+\beta)$ is

$$\frac{l}{r} = \sec\beta\cos(\theta-\alpha) + e\cos(\theta-\psi). \tag{7}$$

4.7 Solved Examples

Example 1. Two conics have a common focus. Then, two of their common chords will pass through the point of intersection of their directrices.

Let the equations of the two conics be

$$\frac{l}{r} = 1 + e\cos\theta \qquad \text{and} \qquad \frac{l'}{r} = 1 + e'\cos(\theta-\alpha).$$

In Cartesian coordinates these equations are

$$(l-ex)^2 - x^2 - y^2 = 0, \tag{1}$$

$$(l' - e'(x\cos\alpha + y\sin\alpha))^2 - x^2 - y^2 = 0. \tag{2}$$

Subtracting (2) from (1) we get

$$(l-ex)^2 - (l' - e'(x\cos\alpha + y\sin\alpha))^2 = 0, \tag{3}$$

which represents some curve through the intersection of (1) and (2). But the equation (3) is a pair of straight lines whose equations are

$$(l-ex) \mp (l' - e'(x\cos\alpha + y\sin\alpha)) = 0$$

or in polar form

$$\left(\frac{l}{r} - e\cos\theta\right) \mp \left(\frac{l'}{r} - e'\cos(\theta-\alpha)\right) = 0\,. \tag{4}$$

Hence two of the chords of intersection of the conics are represented by the equation (4), and these lines clearly pass through the intersection of the directrices whose equations are

$$\frac{l}{r} - e\cos\theta = 0 \qquad \text{and} \qquad \frac{l'}{r} - e'\cos(\theta - \alpha) = 0\,. \qquad \square$$

Example 2. PQ is a chord of an ellipse one of whose foci is S and PQ passes through a fixed point O. Show that

$$\tan\left(\frac{1}{2}\angle PSO\right)\tan\left(\frac{1}{2}\angle QSO\right) = \text{constant.}$$

Solution. Let the ellipse be $l/r = 1 + e\cos\theta$. The equation of a chord PQ of the ellipse, where P and Q have vectorial angles $\alpha - \beta$ and $\alpha + \beta$ respectively, is given by

$$\frac{l}{r} = \sec\beta\cos(\theta - \alpha) + e\cos\theta.$$

This chord passes always through the point $O(\rho, \phi)$, so

$$\frac{l}{\rho} = \sec\beta\cos(\phi - \alpha) + e\cos\phi$$

or

$$\cos(\phi - \alpha) = \left(\frac{l}{\rho} - e\cos\phi\right)\cos\beta \tag{1}$$

Now, $\angle PSO = \phi - (\alpha - \beta)$ and $\angle QSO = (\alpha + \beta) - \phi$. Therefore,

$$\begin{aligned}
&\tan\left(\frac{1}{2}\angle PSO\right)\tan\left(\frac{1}{2}\angle QSO\right)\\
= \ &\tan\left(\frac{\beta + (\phi - \alpha)}{2}\right)\tan\left(\frac{\beta - (\phi - \alpha)}{2}\right)\\
= \ &\frac{\sin^2\left(\frac{\beta}{2}\right) - \sin^2\left(\frac{\phi - \alpha}{2}\right)}{\cos^2\left(\frac{\beta}{2}\right) - \sin^2\left(\frac{\phi - \alpha}{2}\right)} = \frac{\cos(\phi - \alpha) - \cos\beta}{\cos(\phi - \alpha) + \cos\beta}\\
= \ &\frac{\left(\frac{l}{\rho} - e\cos\phi\right)\cos\beta - \cos\beta}{\left(\frac{l}{\rho} - e\cos\phi\right)\cos\beta + \cos\beta} = \frac{\frac{l}{\rho} - e\cos\phi - 1}{\frac{l}{\rho} - e\cos\phi + 1},
\end{aligned}$$

which is constant. □

Example 3. A variable chord PQ of a conic $l/r=1+e\cos\theta$ subtends a constant angle 2α at the focus S. ST is the internal bisector of $\angle PSQ$ which meets the chord in T. Show that the locus of T is the conic

$$\frac{l\cos\alpha}{r}=1+(e\cos\alpha)\cos\theta.$$

Solution. Let the vectorial angles of P and Q be $\beta-\alpha$ and $\beta+\alpha$ so that the angle PSQ is 2α. The chord PQ is

$$\frac{l}{r}=\sec\alpha\cos(\theta-\beta)+e\cos\theta.$$

Let the internal bisector of $\angle PSQ$, which meets the chord in T be ST, and the polar coordinates of T be (ρ,ϕ). Then $\phi=\beta$ and

$$\frac{l}{\rho}=\sec\alpha\cos(\phi-\beta)+e\cos\phi=\sec\alpha+e\cos\phi.$$

Hence locus of T is $l/r=\sec\alpha+e\cos\theta$ or

$$\frac{l\cos\alpha}{r}=1+(e\cos\alpha)\cos\theta.$$ □

Example 4. A chord of a rectangular hyperbola subtends a right angle at the focus. Find the locus of the foot of perpendicular from this focus on the chord.

Solution. Let the equation of the rectangular hyperbola be

$$\frac{l}{r}=1+\sqrt{2}\cos\theta. \tag{1}$$

Let the vectorial angles of extremities of the chord subtending a right angle at the focus of the hyperbola (1) be $\alpha-\pi/4$ and $\alpha+\pi/4$. Then the chord is

$$\frac{l}{r}=\sqrt{2}\cos\theta+\sec\left(\frac{\pi}{4}\right)\cos(\theta-\alpha)=\sqrt{2}\cos\theta+\sqrt{2}\cos(\theta-\alpha). \tag{2}$$

The equation to the line perpendicular to (2) and passing through the focus, the pole, is

$$0 = \sqrt{2}\sin\theta + \sqrt{2}\sin(\theta - \alpha). \tag{3}$$

The foot of the perpendicular is the point of intersection of (2) and (3). Hence the required locus is obtained by eliminating α between (2) and (3). Eliminating α from (2) and (3) we get

$$\left(\frac{l}{r} - \sqrt{2}\cos\theta\right)^2 + \left(\sqrt{2}\sin\theta\right)^2 = 2$$

or

$$\frac{l}{r} = 2\sqrt{2}\cos\theta,$$

which is a straight line. □

4.8 Problem Set 5

1. Assuming the equation of the chord of the conic
$$\frac{l}{r} = 1 + e\cos\theta$$
joining $P(\alpha - \beta)$ and $Q(\alpha + \beta)$ as
$$\frac{l}{r} = a\cos(\theta - \alpha) + b\cos\theta$$
show that the chord is
$$\frac{l}{r} = \sec\beta\,\cos(\theta - \alpha) + e\,\cos\theta\,.$$

2. Assuming the equation of the chord of the conic
$$\frac{l}{r} = 1 + e\cos\theta$$
joining $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ as
$$\begin{vmatrix} 1/r & \cos\theta & \sin\theta \\ 1/r_1 & \cos\theta_1 & \sin\theta_1 \\ 1/r_2 & \cos\theta_2 & \sin\theta_2 \end{vmatrix} = 0$$
show that the chord is
$$\frac{l}{r} = \sec\frac{\theta_2 - \theta_1}{2}\,\cos\left(\theta - \frac{\theta_1 + \theta_2}{2}\right) + e\,\cos\theta\,.$$

3. Find the equation of the chord of the conic $l/r = 1 + e\cos\theta$, joining the points whose vectorial angles are $\pi/6$ and $\pi/2$.

4. If a chord PQ of a conic $l/r = 1 + e\cos\theta$ subtends a right angle at the focus S, then show that
$$\left(\frac{1}{SP} - \frac{1}{l}\right)^2 + \left(\frac{1}{SQ} - \frac{1}{l}\right)^2 = \frac{e^2}{l^2}.$$

5. Find the condition that the chord cut off by the conic $l/r = 1 + e\cos\theta$ from the line $l/r = a\cos\theta + b\sin\theta$ subtends a right angle at the focus.

6. A chord subtends a constant angle 2α at the focus of the conic $l/r = 1 + e\cos\theta$. Prove that, in general, the locus of the foot of perpendicular on it from the focus is the circle
$$(e^2 - \sec^2\alpha)r^2 - 2elr\cos\theta + l^2 = 0.$$
Discuss the case when the conic is a rectangular hyperbola (a hyperbola with eccentricity $\sqrt{2}$) and the constant angle $2\alpha = \pi/2$.

7. PQ is a chord of an ellipse $l/r = 1 + e\cos\theta$ one of whose foci is S and PQ passes through a fixed point $O(\rho, \phi)$. If $k = l/\rho - e\cos\phi$ then show that
$$\tan\left(\frac{1}{2}\angle PSO\right)\tan\left(\frac{1}{2}\angle QSO\right) = \frac{1-k}{1+k}.$$

4.9 Tangent at a point of the conic

Here we obtain the equation of the tangent at a point $T(\alpha)$ on the conic $l/r = 1 + e\cos\theta$.

The tangent at $T(\alpha)$ is the limiting position of the chord joining the points $P(\alpha-\beta)$ and $Q(\alpha+\beta)$ on the conic as $\beta \to 0$. Therefore, taking $\beta = 0$ in the equation of the chord PQ
$$\frac{l}{r} = \sec\beta\,\cos(\theta-\alpha) + e\,\cos\theta$$
we get the equation of the tangent at the point $T(\alpha)$ as

$$\frac{l}{r} = \cos(\theta - \alpha) + e\ \cos\theta = (e + \cos\alpha)\cos\theta + \sin\alpha\sin\theta\,. \qquad (1)$$

Remark. If the given conic is $l/r = 1 + e\cos(\theta - \psi)$ then the tangent at a point $T(\alpha)$ is

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos(\theta - \psi)$$
$$= (e\cos\psi + \cos\alpha)\cos\theta + (e\sin\psi + \sin\alpha)\sin\theta. \qquad (2)$$

If the tangents at the points P and Q on a conic meet in T, then ST bisects the angle PSQ.

4.10 Condition of tangency

Let us find out the condition so that the line $l/r = a\cos\theta + b\sin\theta$ may touch the conic $l/r = 1 + e\cos\theta$.

If the line

$$\frac{l}{r} = a\cos\theta + b\sin\theta,$$

is tangent at a point α on the conic $l/r = 1 + e\cos\theta$, then this line must be same as

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) = (e + \cos\alpha)\cos\theta + \sin\alpha\sin\theta.$$

Identifying the two lines we have

$$a = e + \cos\alpha \qquad \text{and} \qquad b = \sin\alpha.$$

Eliminating α, we get the required condition of tangency as

$$(a - e)^2 + b^2 = 1.$$

Similarly, we can find the condition of tangency of the line

$$\frac{l}{r} = a\cos\theta + b\sin\theta,$$

to the conic

$$\frac{l}{r} = 1 + e\cos(\theta - \gamma)$$

as

$$a^2 + b^2 - 2e(a\cos\gamma + b\sin\gamma) + (e^2 - 1) = 0.$$

4.11 Chord of contact

We find out the equation of chord of contact of tangents from the point $P(\rho, \phi)$ to the conic $l/r = 1 + e\cos\theta$.

Let $\alpha - \beta$ and $\alpha + \beta$ be the vectorial angles of the points of contact Q and R of tangents from $P(\rho, \phi)$. Then the equation of the chord QR is

$$\frac{l}{r} = \sec\beta\cos(\theta - \alpha) + e\cos\theta. \tag{1}$$

Tangents at $Q(\alpha - \beta)$ and $R(\alpha + \beta)$ are given by

$$\frac{l}{r} = e\cos\theta + \cos(\theta - (\alpha - \beta)), \tag{2}$$

$$\frac{l}{r} = e\cos\theta + \cos(\theta - (\alpha + \beta)). \tag{3}$$

Since $P(\rho, \phi)$ lies on (2) and (3), therefore we get

$$\frac{l}{\rho} = e\cos\phi + \cos(\phi - (\alpha - \beta)) = e\cos\phi + \cos(\phi - (\alpha + \beta)). \tag{4}$$

From equation (4) we get

$$\cos(\phi - (\alpha - \beta)) = \cos(\phi - (\alpha + \beta)) \quad \text{or} \quad (\phi - \alpha + \beta) = \pm(\phi - \alpha - \beta).$$

Since $\beta \neq 0$, therefore we get $\phi = \alpha$. Putting $\alpha = \phi$ in the equation (4) we get

$$\cos\beta = \frac{l}{\rho} - e\cos\phi. \tag{5}$$

Using $\alpha = \phi$ and (5) in (1) we get the equation of the chord of contact of tangents from the point $P(\rho, \phi)$ as

$$\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{\rho} - e\cos\phi\right) = \cos(\theta - \phi)\,. \tag{6}$$

4.12 Pair of tangents

Here, we show that the equation of pair of tangents from a point (ρ, ϕ) to $l/r=1+e\cos\theta$ is

$$\left(\left(\frac{l}{r}-e\cos\theta\right)^2-1\right)\left(\left(\frac{l}{\rho}-e\cos\phi\right)^2-1\right)$$

$$= \left(\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{\rho}-e\cos\phi\right)-\cos(\theta-\phi)\right)^2. \tag{1}$$

Let us suppose that the vectorial angle of the point of contact of any one of the tangents drawn from (ρ, ϕ) to the conic $l/r = 1 + e\cos\theta$ is α. The tangent at α is

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha), \tag{2}$$

and this tangent passes through (ρ, ϕ), so that

$$\frac{l}{\rho} = e\cos\phi + \cos(\phi-\alpha). \tag{3}$$

The equation of the pair of tangents will be obtained by eliminating α between (2) and (3). Now, we get

$$\begin{aligned}&\left(\left(\frac{l}{r}-e\cos\theta\right)^2-1\right)\left(\left(\frac{l}{\rho}-e\cos\phi\right)^2-1\right)\\ &\quad= \left(\cos^2(\theta-\alpha)-1\right)\left(\cos^2(\phi-\alpha)-1\right)\\ &\quad= \sin^2(\theta-\alpha)\sin^2(\phi-\alpha).\end{aligned}$$

or

$$\left(\left(\frac{l}{r}-e\cos\theta\right)^2-1\right)\left(\left(\frac{l}{\rho}-e\cos\phi\right)^2-1\right)=\sin^2(\theta-\alpha)\sin^2(\phi-\alpha). \tag{4}$$

On the other hand, we get

$$\begin{aligned}
&\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{\rho} - e\cos\phi\right) - \cos(\theta-\phi) \\
&\quad = \cos(\theta-\alpha)\cos(\phi-\alpha) - \cos(\theta-\phi) \\
&\quad = \frac{1}{2}\left(\cos\left(\theta+\phi-2\alpha\right) + \cos\left(\theta-\phi\right)\right) - \cos(\theta-\phi) \\
&\quad = \frac{1}{2}\left(\cos\left(\theta+\phi-2\alpha\right) - \cos\left(\theta-\phi\right)\right) \\
&\quad = -\ \sin(\theta-\alpha)\sin(\phi-\alpha)
\end{aligned}$$

or

$$\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{\rho} - e\cos\phi\right) - \cos(\theta-\phi) = -\sin(\theta-\alpha)\sin(\phi-\alpha)\,. \tag{5}$$

Hence in view of (4) and (5) the required equation of the pair of tangents is given by (1).

4.13 Asymptotes

4.13.1 Asymptotes from tangents

An asymptote of a conic $l/r = 1 + e\cos\theta$ is a limit of tangent to it, as the point of contact of the tangent tends to infinity, without the tangent itself lying wholly at infinity. The point α is at infinity if $1 + e\cos\alpha = 0$, which implies that

$$\cos\alpha = -\,\frac{1}{e} \quad \text{and} \quad \sin\alpha = \pm\,\sqrt{1-\cos^2\alpha} = \pm\,\frac{\sqrt{e^2-1}}{e}\,.$$

Putting these values in the equation of the tangent

$$\frac{l}{r} = (e + \cos\alpha)\cos\theta + \sin\alpha\sin\theta$$

at the point α, the equations of the asymptotes are

$$\frac{l}{r} = \left(e - \frac{1}{e}\right)\cos\theta \pm \frac{\sqrt{e^2-1}}{e}\sin\theta$$

or,

$$\frac{el}{r} = (e^2-1)\cos\theta \pm \sqrt{e^2-1}\sin\theta\,. \tag{1}$$

4.13.2 Asymptotes from pair of tangents

The asymptotes of a conic $l/r = 1 + e\cos\theta$ are the pair of tangents drawn from the centre of the conic. The coordinates of the centre are $(el/(e^2-1), 0)$. The pair of tangents from a point (ρ, ϕ) to $l/r = 1 + e\cos\theta$ is

$$\left(\left(\frac{l}{r} - e\cos\theta\right)^2 - 1\right)\left(\left(\frac{l}{\rho} - e\cos\phi\right)^2 - 1\right)$$

$$= \left(\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{\rho} - e\cos\phi\right) - \cos(\theta-\phi)\right)^2 .$$

Putting $\rho = el/(e^2-1)$ and $\phi = 0$ in the above equation we get

$$\left(\left(\frac{l}{r} - e\cos\theta\right)^2 - 1\right)\left(\left(\frac{e^2-1}{e} - e\right)^2 - 1\right)$$

$$= \left(\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{e^2-1}{e} - e\right) - \cos\theta\right)^2 ,$$

or

$$\left(\frac{l^2}{r^2} - \frac{2el}{r}\cos\theta + e^2\cos^2\theta - 1\right)\left(\frac{1-e^2}{e^2}\right) = \frac{l^2}{e^2r^2} ,$$

or

$$\left(\frac{l^2}{r^2} - \frac{2el}{r}\cos\theta + e^2\cos^2\theta - 1\right)(1-e^2) = \frac{l^2}{r^2} ,$$

or

$$\frac{e^2l^2}{r^2} + \frac{2el}{r}(1-e^2)\cos\theta - e^2(1-e^2)\cos^2\theta + (1-e^2) = 0\,,$$

or

$$\frac{e^2l^2}{r^2} + \frac{2el}{r}(1-e^2)\cos\theta + (1-e^2)^2\cos^2\theta$$

$$= e^2(1-e^2)\cos^2\theta - (1-e^2) + e^2(1-e^2)\cos^2\theta\,,$$

or

$$\left(\frac{el}{r} + (1 - e^2)\cos\theta\right)^2 = (e^2 - 1)\sin\theta\,,$$

or

$$\frac{el}{r} = (e^2 - 1)\cos\theta \pm \sqrt{e^2 - 1}\sin\theta\,.$$

4.14 Solved Examples

Example 1. Let PSQ be a focal chord of the conic $l/r = 1 + e\cos\theta$. Then show that
(a) the tangents at P and Q intersect on the corresponding directrix,
(b) the angle between the tangents at P and Q is $\arctan\left(\dfrac{2e\sin\alpha}{1-e^2}\right)$ where α is the angle between chord and the axis of the conic.

Solution. **(a)** The tangents at $P(\alpha)$ and $Q(\alpha + \pi)$ are

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$$

and

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha - \pi) = e\cos\theta - \cos(\theta - \alpha).$$

If these tangents intersect in a point $T(\rho, \phi)$ then substituting these values in the two equations and eliminating α from them, we get $l/\rho = e\cos\phi$, which shows that T lies on the directrix $l/r = e\cos\theta$.

(b) The tangent at $P(\alpha)$ may be written as

$$\frac{l}{r} = (e + \cos\alpha)\cos\theta + \sin\alpha\sin\theta = a\cos(\theta - \phi),$$

where

$$\tan\phi = \frac{\sin\alpha}{e + \cos\alpha}.$$

Hence the perpendicular from the focus on the tangent at $P(\alpha)$ makes an angle

$$\arctan\left(\frac{\sin\alpha}{e + \cos\alpha}\right)$$

with the polar axis. Similarly, the perpendicular from the focus on the tangent at $Q(\alpha+\pi)$ makes an angle

$$\arctan\left(\frac{-\sin\alpha}{e-\cos\alpha}\right)$$

with the polar axis. Hence the angle between the tangents at P and Q is

$$\begin{aligned}
& \arctan\left(\frac{\sin\alpha}{e+\cos\alpha}\right) - \arctan\left(\frac{-\sin\alpha}{e-\cos\alpha}\right) \\
= \quad & \arctan\left(\frac{\dfrac{\sin\alpha}{e+\cos\alpha} - \dfrac{-\sin\alpha}{e-\cos\alpha}}{1+\left(\dfrac{\sin\alpha}{e+\cos\alpha}\right)\left(\dfrac{-\sin\alpha}{e-\cos\alpha}\right)}\right) \\
= \quad & \arctan\left(\frac{2e\sin\alpha}{1-e^2}\right).
\end{aligned}$$

Example 2. (**Auxiliary circle**) Prove that the equation to the locus of the foot of perpendicular from focus of the conic $l/r = 1+e\cos\theta$ on a tangent to it is

$$(1-e^2)r^2 + 2elr\cos\theta - l^2 = 0\,.$$

Discuss the particular case when $e = 1$.

Solution. The equation of tangent at α on the conic $l/r = 1+e\cos\theta$ is

$$\frac{l}{r} = \cos(\theta-\alpha) + e\cos\theta. \tag{1}$$

The equation to the line perpendicular to (1) and passing through the focus, the pole, is

$$0 = \sin(\theta-\alpha) + e\sin\theta. \tag{2}$$

The foot of the perpendicular is the point of intersection of (1) and (2). Hence the required locus is obtained by eliminating α between (1) and (2). From (1) and (2) we get respectively

$$\cos(\theta-\alpha) = \frac{l}{r} - e\cos\theta \qquad \text{and} \qquad \sin(\theta-\alpha) = -e\sin\theta.$$

Squaring and adding we get

$$1 = \left(\frac{l}{r} - e\cos\theta\right)^2 + e^2\sin^2\theta = \frac{l^2}{r^2} - \frac{2el}{r}\cos\theta + e^2$$

or

$$r^2(1-e^2) + 2elr\cos\theta - l^2 = 0.$$

This locus represents a circle, when $1-e^2 \neq 0$, that is when the conic is not a parabola. This circle is known as the auxiliary circle of the conic. When $e=1$, the locus becomes

$$\frac{l}{r} \;=\; 2\cos\theta \;=\; \cos(\theta-0)+\cos\theta,$$

which is the equation of tangent at the vertex of the parabola. □

Example 3. (**Director circle**) Obtain the locus of the point of intersection of perpendicular tangents of the conic $l/r = 1 + e\cos\theta$ as

$$r^2(1-e^2) + 2elr\cos\theta - 2l^2 = 0.$$

Solution. Let the two tangents be

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha) = (e+\cos\alpha)\cos\theta + \sin\alpha\sin\theta, \tag{1}$$

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\beta) = (e+\cos\beta)\cos\theta + \sin\beta\sin\theta. \tag{2}$$

The tangents will meet where

$$\theta = \frac{1}{2}(\alpha+\beta) \qquad \text{and} \qquad \frac{l}{r} = e\cos\theta + \cos\frac{1}{2}(\alpha-\beta). \tag{3}$$

From (1) and (2) we see that the condition of perpendicularity is

$$(e+\cos\alpha)(e+\cos\beta) + \sin\alpha\sin\beta = 0 \text{ or}$$

$$e^2 + 2e\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta) + 2\cos^2\frac{1}{2}(\alpha-\beta) - 1 = 0.$$

Hence using the equation (3) in the above equation, we get

$$e^2 - 1 + 2e\cos\theta\left(\frac{l}{r} - e\cos\theta\right) + 2\left(\frac{l}{r} - e\cos\theta\right)^2 = 0 \text{ or}$$

$$r^2(1-e^2) + 2elr\cos\theta - 2l^2 = 0,$$

which is the required locus. It is a circle if $e \neq 1$. This circle is called the director circle. If $e = 1$, the conic is a parabola and the above equation reduces to

$$2elr\cos\theta - 2l^2 = 0 \quad \text{or} \quad l = r\cos\theta\,,$$

which is the directrix of the parabola. Hence the locus of the point of intersection of perpendicular tangents to a parabola is its directrix. □

Example 4. Two conics have a common focus and directrix. If any tangent to one intersects the other in P and Q, show that

$$\sec\left(\frac{1}{2}\angle PSQ\right) = \frac{e'}{e}.$$

Solution. Let the two conics be

$$\frac{ed}{r} = 1 + e\cos\theta\,, \tag{1}$$

$$\frac{e'd}{r} = 1 + e'\cos\theta\,, \tag{2}$$

Let the tangent at a point γ to the conic (1)

$$\frac{ed}{r} = (e + \cos\gamma)\cos\theta + \sin\gamma\sin\theta \tag{3}$$

intersects the conic (2) in the points $P(\alpha-\beta)$ and $Q(\alpha+\beta)$. Then the tangent (3) is identical with the chord PQ

$$\frac{e'd}{r} = (e' + \cos\alpha\sec\beta)\cos\theta + \sin\alpha\sec\beta\sin\theta \tag{4}$$

of the conic (2). Therefore

$$\frac{e}{e'} = \frac{e+\cos\gamma}{e'+\cos\alpha\sec\beta} = \frac{\sin\gamma}{\sin\alpha\sec\beta}\,,$$

which implies that

$$e'\cos\gamma = e\sec\beta\cos\alpha \quad \text{and} \quad e'\sin\gamma = e\sec\beta\sin\alpha\,.$$

Squaring and adding we get

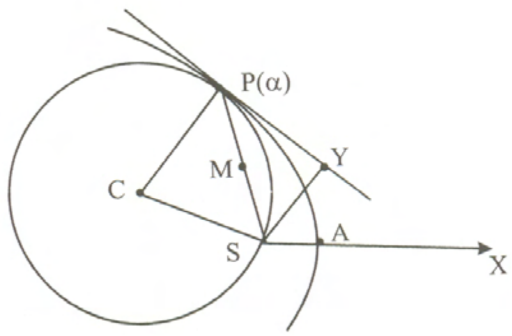
$$e'^2 = e^2 \sec^2 \beta \quad \text{or} \quad \frac{e'}{e} = \sec\beta = \sec\left(\frac{1}{2}\angle PSQ\right),$$

since $\angle PSQ = (\alpha+\beta) - (\alpha-\beta) = 2\beta$. □

Example 5. A circle is drawn through the focus of the parabola $2a/r = 1+\cos\theta$ to touch it at the point $\theta = \alpha$. Show that its equation is

$$r\cos^3\alpha/2 = a\cos(\theta - 3\alpha/2).$$

Solution. The tangent at the point $P(\alpha)$ is drawn. Y is a point on this tangent, such that SY is perpendicular to the tangent at P. The point M is the mid-point of SP and the point C lies on the normal at P such that $SC = CP$. Thus the point C is the centre of the required circle.



The tangent at $P(\alpha)$ to the parabola is

$$\frac{2a}{r} = \cos\theta + \cos(\theta-\alpha) = 2\cos\alpha/2\cos(\theta-\alpha/2)$$

or

$$a = r\cos\alpha/2\cos(\theta-\alpha/2)\,.$$

Hence the perpendicular SY to the tangent at P makes an angle $\alpha/2$ with the initial line; and

$$\angle YSP = \alpha - \alpha/2 = \alpha/2 \qquad \text{or} \qquad \angle MPC = \alpha/2\,.$$

If b is the radius of the required circle,then $MP/b = \cos\alpha/2$ so $SP/2 = MP = b\cos\alpha/2$, which gives

$$\frac{1}{2}\left(\frac{2a}{1+\cos\alpha}\right) = b\cos\frac{\alpha}{2} \quad \text{or} \quad b = \frac{a}{2\cos^2\alpha/2\cos\alpha/2}\,.$$

Also, $\angle CSx = \alpha + \angle MSC = \alpha + \alpha/2 = 3\alpha/2$. Thus the equation of required circle is

$$r = 2b\cos(\theta - 3\alpha/2) = \frac{a}{\cos^3\alpha/2}\cos(\theta - 3\alpha/2)$$

or

$$r\cos^3\alpha/2 = a\cos(\theta - 3\alpha/2)\,. \qquad \square$$

Example 6. A conic is described having the same focus and eccentricity as the conic $l/r = 1 + e\cos\theta$ and the two conics touch at the point $\theta = \alpha$. Prove that the length of its latus rectum is

$$\frac{2l(1-e^2)}{e^2 + 2e\cos\alpha + 1}\,,$$

and that the angle between their axis is

$$2\arctan\left(-\frac{e+\cos\alpha}{\sin\alpha}\right)\,.$$

Solution. Let the two conics be

$$\frac{l}{r} = 1 + e\cos\theta, \tag{1}$$

$$\frac{k}{r} = 1 + e\cos(\theta - \gamma). \tag{2}$$

These conics touch at the point whose vectorial angle is α. The tangents at the point of contact α on the conics are

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) = (e + \cos\alpha)\cos\theta + \sin\alpha\sin\theta, \tag{3}$$

$$\begin{aligned}\frac{k}{r} &= e\cos(\theta-\gamma)+\cos(\theta-\alpha)\\ &= (e\cos\gamma+\cos\alpha)\cos\theta+(e\sin\gamma+\sin\alpha)\sin\theta. \qquad (4)\end{aligned}$$

Since (3) and (4) are identical, therefore

$$\frac{k}{l}=\frac{e\cos\gamma+\cos\alpha}{e+\cos\alpha}=\frac{e\sin\gamma+\sin\alpha}{\sin\alpha},$$

which implies that

$$e\cos\gamma=\left(\frac{k}{l}-1\right)\cos\alpha+\frac{ek}{l} \qquad (5)$$

and

$$e\sin\gamma=\left(\frac{k}{l}-1\right)\sin\alpha. \qquad (6)$$

Squaring and adding we get

$$\frac{k^2}{l^2}(e^2+2e\cos\alpha+1)-2\frac{k}{l}(1+e\cos\alpha)+(1-e^2)=0,$$

which implies that

$$\frac{k}{l}=1\ ,\ \frac{1-e^2}{e^2+2e\cos\alpha+1}.$$

Dividing (6) by (5) and putting $k=l(1-e^2)/(e^2+2e\cos\alpha+1)$ we get

$$\begin{aligned}\gamma &= \arctan\left(\frac{(k-l)\sin\alpha}{(k-l)\cos\alpha+ke}\right)\\ &= \arctan\left(\frac{(l-le^2-le^2-2el\cos\alpha-l)\sin\alpha}{(l-le^2-le^2-2el\cos\alpha-l)\cos\alpha+le-le^3}\right)\\ &= \arctan\left(\frac{le(-2e-2\cos\alpha)\sin\alpha}{le(-2e-2\cos\alpha)\cos\alpha+1-e^2}\right)\\ &= \arctan\left(\frac{\left(-\frac{e+\cos\alpha}{\sin\alpha}\right)+\left(-\frac{e+\cos\alpha}{\sin\alpha}\right)}{1-\left(-\frac{e+\cos\alpha}{\sin\alpha}\right)\left(-\frac{e+\cos\alpha}{\sin\alpha}\right)}\right)\\ &= 2\arctan\left(-\frac{e+\cos\alpha}{\sin\alpha}\right).\end{aligned}$$

Example 7. If A, B, C are any three points on a parabola, and the tangents at these points form a triangle $A'B'C'$, then prove that $SA \cdot SB \cdot SC = SA' \cdot SB' \cdot SC'$, where S is the focus of the parabola.

Solution. Let $A\left(\frac{l}{2}\sec^2\alpha, 2\alpha\right)$, $B\left(\frac{l}{2}\sec^2\beta, 2\beta\right)$ and $C\left(\frac{l}{2}\sec^2\gamma, 2\gamma\right)$ be three points on a parabola $l/r = 1+\cos\theta$. Then

$$SA \cdot SB \cdot SC = (l^3/8)\sec^2\alpha\sec^2\beta\sec^2\gamma.$$

The equations of tangents at A, B and C are

$$\frac{l}{r} = \cos\theta + \cos(\theta - 2\alpha),$$

$$\frac{l}{r} = \cos\theta + \cos(\theta - 2\beta)$$

and

$$\frac{l}{r} = \cos\theta + \cos(\theta - 2\gamma)$$

respectively. Hence at A' we get

$$\theta = \beta + \gamma, \quad r = \frac{l}{2}\sec\beta\sec\gamma\,;$$

at B' we get

$$\theta = \gamma + \alpha, \quad r = \frac{l}{2}\sec\gamma\sec\alpha\,;$$

and at C' we get

$$\theta = \alpha + \beta, \quad r = \frac{l}{2}\sec\alpha\sec\beta.$$

Hence

$$SA' \cdot SB' \cdot SC' = \frac{l^3}{8}\sec^2\alpha\sec^2\beta\sec^2\gamma.$$

Thus we get

$$SA \cdot SB \cdot SC = \frac{l^3}{8}\sec^2\alpha\sec^2\beta\sec^2\gamma = SA' \cdot SB' \cdot SC'. \qquad \square$$

Example 8. If PQ is the chord of contact of tangents drawn from a point T to a parabola with focus S, then $SP \cdot SQ = ST^2$.

Let $l/r = 1 + \cos\theta$ be the given parabola and let the vectorial angles of P and Q be $\alpha-\beta$ and $\alpha+\beta$ respectively, so that $\angle PSQ = 2\beta$. Tangents at P and Q are

$$\frac{l}{r} = \cos\theta + \cos(\theta - \alpha + \beta) \quad \text{and} \quad \frac{l}{r} = \cos\theta + \cos(\theta - \alpha - \beta).$$

These tangents intersect in the point $T(l/(\cos\alpha + \cos\beta), \alpha)$. Now

$$\begin{aligned}
\frac{l}{SP} \cdot \frac{l}{SQ} &= (1 + \cos(\alpha - \beta))(1 + \cos(\alpha + \beta)) \\
&= 1 + (\cos(\alpha + \beta) + \cos(\alpha - \beta)) \\
&\qquad + \cos(\alpha - \beta)\cos(\alpha + \beta) \\
&= 1 + 2\cos\alpha\cos\beta + \frac{1}{2}(\cos 2\alpha + \cos 2\beta) \\
&= 1 + 2\cos\alpha\cos\beta + \frac{1}{2}(2\cos^2\alpha - 1 + 2\cos^2\beta - 1) \\
&= (\cos\alpha + \cos\beta)^2 = \frac{l^2}{ST^2}.
\end{aligned}$$

Example 9. A variable chord PQ of a conic $l/r = 1 + e\cos\theta$ subtends a constant angle 2β at the focus S. Show that the locus of the pole of PQ is the conic

$$\frac{l\sec\beta}{r} = 1 + (e\sec\beta)\cos\theta.$$

Solution. Let PQ be a chord subtending a constant angle 2β at the focus S and let $T(\rho, \phi)$ be the point of intersection of tangents at P and Q. Then T is the pole of the chord PQ. The vectorial angles of P and Q can be taken as $\alpha - \beta$ and $\alpha + \beta$ so that the angle $\angle PSQ$ is 2β. The equations of tangents at P and Q are

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha + \beta) \quad \text{and} \quad \frac{l}{r} = e\cos\theta + \cos(\theta - \alpha - \beta).$$

Since the point $T(\rho, \phi)$ lies on both the tangents, therefore we have

$$\frac{l}{\rho} = e\cos\phi + \cos(\phi - \alpha + \beta)\,, \tag{1}$$

$$\frac{l}{\rho} = e\cos\phi + \cos(\phi - \alpha - \beta). \tag{2}$$

From (1) and (2) we see that $\cos(\phi - \alpha + \beta) = \cos(\phi - \alpha - \beta)$ or $\phi - \alpha + \beta = \pm(\phi - \alpha - \beta)$. Taking the negative sign, we get $\phi = \alpha$. Substituting this value of α in (1), we get

$$\frac{l}{\rho} = e\cos\phi + \cos\beta \quad \text{or} \quad \frac{l\sec\beta}{\rho} = 1 + (e\sec\beta)\cos\phi.$$

Hence the locus of the point $T(\rho, \phi)$ is

$$\frac{l\sec\beta}{r} = 1 + (e\sec\beta)\cos\theta.$$

We see that this conic with eccentricity $e\sec\beta$ has the same focus and directrix as of the conic given in the problem. Moreover, this conic is an ellipse, a parabola or a hyperbola according as $e\sec\beta$ is less than, equal to or greater than 1; or $\cos\beta$ is less than, equal to or greater than e. □

Example 10. Given the focus and the directrix of a conic, show that the polar of a given point with respect to it passes through a fixed point.

Solution. The conics may be taken as

$$\frac{de}{r} = 1 + e\cos\theta, \tag{1}$$

where the focus is the pole and directrix is $d = r\cos\theta$. Let the given point be $P(\rho, \phi)$. The polar of $P(\rho, \phi)$ to the conic (1) is

$$\left(\frac{de}{r} - e\cos\theta\right)\left(\frac{de}{\rho} - e\cos\phi\right) = \cos(\theta - \phi),$$

or

$$e^2\left(\frac{d}{r} - \cos\theta\right)\left(\frac{d}{\rho} - \cos\phi\right) = \cos(\theta - \phi). \tag{2}$$

The line passes through a fixed point, if the coefficient of e^2 in (2) is zero, which implies that

$$\cos(\theta - \phi) = 0 \qquad \text{or} \qquad \theta = \phi + \frac{\pi}{2}.$$

Hence $d/r = \cos\theta$ implies that $r = d/\cos(\phi + \pi/2) = -d\csc\phi$. Thus the fixed point is $(-d\csc\phi, \phi + \pi/2)$. □

Example 11. A circle has centre at one focus of a hyperbola, and radius equal to a quarter of the latus rectum of the hyperbola. Then the straight lines joining the focus to the points of intersection of the circle and the hyperbola are parallel to the asymptotes of the hyperbola.

Let the hyperbola be

$$\frac{l}{r} = 1 + e\cos\theta. \tag{1}$$

Asymptotes to (1) are

$$\frac{el}{r} = (e^2 - 1)\cos\theta \pm \sqrt{e^2 - 1}\sin\theta\,. \tag{2}$$

In Cartesian coordinates (2) is

$$el = (e^2 - 1)x \pm \sqrt{e^2 - 1}y\,,$$

which implies that the slope of (2) is

$$\tan\theta = \pm\sqrt{e^2 - 1} \qquad \text{or} \qquad \theta = \pm\cos^{-1}\left(\frac{1}{e}\right).$$

On the other hand, the circle $r = l/2$ intersects (1) in the points whose vectorial angles are given by

$$\frac{l}{l/2} = 1 + e\cos\theta \quad \text{or} \quad e\cos\theta = 1 \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{1}{e}\right).$$

4.15 Problem Set 6

1. In any conic prove that the portion of tangent intercepted between the conic and the directrix subtends a right angle at the corresponding focus.

2. Prove that the angle between the tangents at the points α and β on a parabola $l/r = 1 + \cos\theta$ is $(\beta - \alpha)/2$.

3. If the tangent at the extremities of a focal chord meet the transverse axis in T and T', then prove that $1/ST + 1/ST'$ is a constant.

4. The tangents to a conic from a variable point on the latus rectum are produced. Show that the sum of the reciprocals of the focal distances of their points of contact is a constant.

5. Let the tangents at the points P and Q of the conic $l/r = 1 + e\cos\theta$ meet in T and let PQ subtend a constant angle 2β at the focus S. Then show that

$$\frac{1}{SP} + \frac{1}{SQ} - \frac{2\cos\beta}{ST}$$

is a constant.

6. Obtain the polar equation of the circle circumscribing the triangle formed by the tangents to the parabola $l/r = 1+\cos\theta$ at the points whose vectorial angles are 2α, 2β, 2γ in the form

$$2r\cos\alpha\cos\beta\cos\gamma = l\cos(\theta - \alpha - \beta - \gamma)\,.$$

7. Obtain the polar equation of the circle circumscribing the triangle formed by the tangents to the parabola $2a/r = 1 - \cos\theta$ at the points whose vectorial angles are α, β, γ in the form

$$r = a\csc\frac{\alpha}{2}\csc\frac{\beta}{2}\csc\frac{\gamma}{2}\sin\left(\frac{\alpha+\beta+\gamma}{2} - \theta\right)\,.$$

8. If tangents are drawn to the parabola $2a/r = 1 - \cos\theta$ at points whose vectorial angles are α, β, γ, δ, then show that the centres of the circle circumscribing the four triangles formed by these four lines all lie on the circle whose equation is

$$r = -\frac{a}{2}\csc\frac{\alpha}{2}\csc\frac{\beta}{2}\csc\frac{\gamma}{2}\csc\frac{\delta}{2}\cos\left(\theta - \frac{\alpha+\beta+\gamma+\delta}{2}\right).$$

9. Let a circle, passing through the focus of a given parabola, be given. Let a conic with latus rectum $2l$ and with the same focus be given such that its directrix touches the parabola. If the circle intersects the conic in four points whose distances from the focus are r_1, r_2, r_3 and r_4, then show that $r_1r_2r_3r_4$ is a constant.

10. Show that the equation of the circle, which passes through the focus of the conic $l/r = 1 + e\cos\theta$ and touches it at the point $\theta = \alpha$ is

$$r(1 + e\cos\alpha)^2 = l\cos(\theta - \alpha) + el\cos(\theta - 2\alpha)$$

11. Show that the line $p = r\cos(\theta - \alpha)$ touches the parabola $l/r = 1 + \cos\theta$ if and only if $p = (l/2)\sec\alpha$.

12. Prove that the conics $l_1/r = 1 + e_1\cos\theta$ and $l_2/r = 1 + e_2\cos(\theta - \alpha)$ will touch one another if

$${l_1}^2(1 - {e_2}^2) + {l_2}^2(1 - {e_1}^2) = 2l_1l_2(1 - e_1e_2\cos\alpha).$$

13. Show that the conics $2l\sqrt{3} = r(\sqrt{3} + \cos\theta)$ and $l\sqrt{3} = r(\sqrt{3} + \cos(\theta + \pi/3)$ touch where $\theta = \pi/2$.

14. P, Q, R are three points on the conic $l/r = 1 + e\cos\theta$, the focus S being the pole, SP and SR meet the tangent at Q in M and N so that $SM = SN = l$. Prove that PR touches $l/r = 1 + 2e\cos\theta$.

15. Let PP' and QQ' be two focal chords of a parabola. Let the tangent at P meets the tangents at Q and Q' in the points N

and N' respectively and the tangent at P' meets the tangents at Q and Q' in the points K' and K respectively. Then show that the lines NK and $N'K'$ pass through the focus and are at right angles.

16. Show that the locus of point of intersection of two tangents to the parabola $l/r = 1 + \cos\theta$, which cut each other at constant angle α is a conic whose latus rectum is $2l \sec\alpha$ and eccentricity $\sec\alpha$, that is
$$\frac{l \sec\alpha}{r} = 1 + \sec\alpha \cos\theta.$$

17. A variable chord PQ of a conic $l/r = 1 + e\cos\theta$ subtends a constant angle 2β at the focus S. Show that PQ always touches a conic having the same focus and directrix.

18. Prove that the chords of a rectangular hyperbola which subtend a right angle at a focus touch a fixed parabola.

19. If the tangents at two points P and Q of a conic meet in a point T and if the straight line PQ meets the directrix corresponding to the focus S in a point K, then prove that $\angle KST$ is a right angle.

20. Let S be the focus of a given conic. Let T be a fixed point. If SK is drawn perpendicular to ST to meet the directrix in K, then show that the polar of T will pass through the fixed point K.

21. Show that two points having the vectorial angles α and β on the conic $l/r = 1 + e\cos\theta$ will be the ends of a diameter if
$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{e+1}{e-1}.$$

22. Two equal ellipses of eccentricity e are placed with their axes at right angles and have a common focus S. If PQ is a common tangent to the two ellipses then prove that the $\angle PSQ = 2\sin^{-1}(e/\sqrt{2})$.

23. Chords PQ and PR subtend equal angles at a focus of a conic. Prove that the chord QR and the tangent at P meet on the directrix corresponding to the focus.

24. Prove that two equal conics which have a common focus and whose axes are inclined at an angle 2α to one another; intersect at an angle

$$\arctan\left(\frac{e^2\sin 2\alpha + 2e\sin\alpha}{e^2\cos 2\alpha + 2e\cos\alpha + 1}\right).$$

25. Prove that the line $l/r = A\cos\theta + B\sin\theta$, will touch the conic $l/r = 1 + e\cos(\theta - \gamma)$ if $A^2 + B^2 - 2e(A\cos\gamma + B\sin\gamma) + (e^2 - 1) = 0$.

26. Tangents are drawn at the extremities of perpendicular focal radii of a conic $l/r = 1 + e\cos\theta$. Show that the locus of their point of intersection is another conic having the same focus.

27. If PQ is the chord of contact of tangents drawn from a point T to a conic $l/r = 1 + e\cos\theta$, whose focus is S, then prove that

$$\frac{1}{SP \cdot SQ} - \frac{1}{ST^2} = \frac{1 - e^2}{l^2}\sin^2\left(\frac{1}{2}\angle PSQ\right).$$

Hence deduce that, if the conic is a parabola, then $SP \cdot SQ = ST^2$.

28. Show that the locus of point of intersection of two tangents to the parabola $l/r = 1 + \cos\theta$, which cut another at constant angle α is a conic whose latus rectum is $2l\sec\alpha$ and eccentricity $\sec\alpha$, that is,

$$\frac{l\sec\alpha}{r} = 1 + \sec\alpha\cos\theta.$$

29. Show that the polar equation of a line cutting the conic $l/r = 1 - e\cos\theta$ in points $(\pi/4 - \alpha)$ and $(\pi/4 + \alpha)$ is

$$\frac{l}{r}\cos\alpha = \cos(\theta - \pi/4) - e\cos\theta\cos\alpha.$$

Hence deduce that the polar equation of the tangent to the conic at the point whose vectorial is $\pi/4$.

30. Prove that the tangents from a point to a conic subtend angles at a focus, which are equal or supplementary.

31. Two parabolas have a common focus and their axes are inclined at an angle 2α to one another. Prove that the locus of the point of intersection of perpendicular tangents one to each is a conic.

4.16 Normal at a point of the conic

Let $l/r = 1 + e\cos(\theta - \psi)$ be a conic. The tangent to the conic at a point $T\left(\dfrac{l}{1+e\cos(\alpha-\psi)}, \alpha\right)$ is

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos(\theta - \psi).$$

The normal at the point T on the conic is perpendicular to the tangent at T. Therefore, its equation can be taken as

$$\begin{aligned}\frac{l'}{r} &= \cos\left(\theta - \alpha + \frac{\pi}{2}\right) + e\cos\left(\theta - \psi + \frac{\pi}{2}\right)\\ &= -\sin(\theta - \alpha) - e\sin(\theta - \psi),\end{aligned}$$

where l' is to be determined so that the point T lies on it. Since the point T lies on the above normal, therefore

$$\left(\frac{1+e\cos(\alpha-\psi)}{l}\right) l' = -e\sin(\alpha-\psi) \text{ or } l' = \left(\frac{-e\sin(\alpha-\psi)}{1+e\cos(\alpha-\psi)}\right) l.$$

Hence, equation of the normal at the point $T\,(l/\,(1+e\cos(\alpha-\psi))\,,\alpha)$ on the conic $l/r = 1 + e\cos(\theta - \psi)$ is

$$\left(\frac{e\sin(\alpha-\psi)}{1+e\cos(\alpha-\psi)}\right)\frac{l}{r} = \sin(\theta-\alpha) + e\sin(\theta-\psi). \tag{1}$$

Particular cases. If the given conic is $l/r = 1 + e\cos\theta$, then the normal at a point $T(\alpha)$ is

$$\left(\frac{e\sin\alpha}{1+e\cos\alpha}\right)\frac{l}{r} = \sin(\theta-\alpha) + e\sin\theta. \tag{2}$$

If the given conic is $l/r = 1 - e\cos\theta$, then the normal at a point $T(\alpha)$ is

$$\left(\frac{e\sin\alpha}{1-e\cos\alpha}\right)\frac{l}{r} = e\sin\theta - \sin(\theta-\alpha). \tag{3}$$

4.17 Solved Examples

Example 1. Normal at $(l, \pi/2)$ of $l/r = 1+e\cos\theta$ meets the conic at Q. Show that

$$SQ = \frac{l(1+3e^2+e^4)}{1+e^2-e^4}.$$

Solution. The given conic is

$$\frac{l}{r} = 1 + e\cos\theta\,. \tag{1}$$

A normal to the conic (1) at the point $(\pi/2)$ is

$$\left(\frac{e\sin(\pi/2)}{1+e\cos(\pi/2)}\right)\frac{l}{r} = e\sin\theta + \sin(\theta-\pi/2)$$

or

$$\frac{el}{r} = e\sin\theta - \cos\theta\,. \tag{2}$$

The normal (2) intersects the conic (1) in the point Q whose vectorial angle satisfies

$$e(1+e\cos\theta) = e\sin\theta - \cos\theta\,,$$

which gives

$$e + (e^2+1)\cos\theta = e\sin\theta\,.$$

Squaring the above equation we get

$$e^2 + (e^2+1)^2\cos^2\theta + 2e(e^2+1)\cos\theta = e^2\sin^2\theta\,,$$

or

$$e^2(1-\sin^2\theta) + (e^2+1)^2\cos^2\theta + 2e(e^2+1)\cos\theta = 0\,,$$

which implies that

$$\cos\theta = 0 \qquad \text{or} \qquad \cos\theta = -\frac{2e(e^2+1)}{e^4+3e^2+1}\,.$$

Putting the last value of $\cos\theta$ in the equation (1) we get

$$\frac{l}{SQ} = 1 + e\left(\frac{-2e(e^2+1)}{e^4+3e^3+1}\right) \quad \text{or} \quad SQ = \frac{l(1+3e^2+e^4)}{1+e^2-e^4}.$$

Example 2. If the normal to the conic $l/r = 1 + e\cos\theta$ at a point α meets the conic at β again, then prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = -\frac{1+2e\cos^2\dfrac{\alpha}{2}+e^2}{1-2e\sin^2\dfrac{\alpha}{2}+e^2}\,.$$

Solution. A normal to the conic at a point α is

$$\left(\frac{e\sin\alpha}{1+e\cos\alpha}\right)\frac{l}{r} = \sin(\theta-\alpha) + e\sin\theta\,.$$

This normal meets the conic in the point $(l/(1+e\cos\beta), \beta)$ if

$$\frac{e\sin\alpha(1+e\cos\beta)}{(1+e\cos\alpha)} = \sin(\beta-\alpha) + e\sin\beta$$

or

$$\begin{aligned} 0 &= \sin(\alpha-\beta)(1+e\cos\alpha+e^2) + e(\sin\alpha - \sin\beta) \\ &= 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}(1+e\cos\alpha+e^2) \\ &\quad + 2e\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \end{aligned}$$

or

$$\begin{aligned} 0 &= \cos\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)(1+e\cos\alpha+e^2) + e\cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) \\ &= \left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)(1+e\cos\alpha+e^2) \\ &\quad + e\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) \end{aligned}$$

or

$$0=\left(1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)(1+e\cos\alpha+e^2)+e\left(1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$$

or

$$0=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}(1+e\cos\alpha+e^2-e)+(1+e\cos\alpha+e^2+e)\,,$$

or

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=-\frac{1+2e\cos^2\dfrac{\alpha}{2}+e^2}{1-2e\sin^2\dfrac{\alpha}{2}+e^2}\,. \qquad \square$$

Example 3. If the normals at three points α, β, γ on the parabola $l/r=1+\cos(\theta-\psi)$ meet in a point (ρ,ϕ), then show that $2\phi=\alpha+\beta+\gamma-\psi$.

Solution. The parabola is

$$\frac{l}{r}=1+\cos(\theta-\psi)\,. \tag{1}$$

A normal to (1) is

$$\left(\frac{\sin(\lambda-\psi)}{1+\cos(\lambda-\psi)}\right)\frac{l}{r}=\sin(\theta-\lambda)+\sin(\theta-\psi)\,. \tag{2}$$

If (2) passes through a point (ρ,ϕ), then

$$\begin{aligned}\left(\frac{\sin(\lambda-\psi)}{1+\cos(\lambda-\psi)}\right)\frac{l}{r} &= \sin(\phi-\lambda)+\sin(\phi-\psi)\\ &= \sin((\phi-\psi)-(\lambda-\psi))+\sin(\phi-\psi)\\ &= (1+\cos(\lambda-\psi))\sin(\phi-\psi)\\ &\qquad -\ \sin(\lambda-\psi)\cos(\phi-\psi)\,.\end{aligned}$$

Using

$$\sin(\lambda-\psi)=\frac{2\tan\dfrac{(\lambda-\psi)}{2}}{1+\tan^2\dfrac{(\lambda-\psi)}{2}}$$

and

$$\cos(\lambda-\psi)=\frac{1-\tan^2\dfrac{(\lambda-\psi)}{2}}{1+\tan^2\dfrac{(\lambda-\psi)}{2}}$$

we get

$$\left(\frac{2\tan\dfrac{(\lambda-\psi)}{2}}{2}\right)\frac{l}{\rho} = \left(\frac{2}{1+\tan^2\dfrac{(\lambda-\psi)}{2}}\right)\sin(\phi-\psi) - \left(\frac{2\tan\dfrac{(\lambda-\psi)}{2}}{1+\tan^2\dfrac{(\lambda-\psi)}{2}}\right)\cos(\phi-\psi)$$

or

$$l\left(\tan\frac{(\lambda-\psi)}{2}\right)\left(1+\tan^2\frac{(\lambda-\psi)}{2}\right) = 2\rho\sin(\phi-\psi) - 2\rho\tan\frac{(\lambda-\psi)}{2}\cos(\phi-\psi)$$

or

$$l\tan^3\frac{(\lambda-\psi)}{2} + (l+2\rho\cos(\phi-\psi))\tan\frac{(\lambda-\psi)}{2} - 2\rho\sin(\phi-\psi) = 0. \quad (3)$$

Thus the roots of this equation are $\tan\left(\frac{\alpha-\psi}{2}\right)$, $\tan\left(\frac{\beta-\psi}{2}\right)$ and $\tan\left(\frac{\gamma-\psi}{2}\right)$; and we get

$$\tan\left(\tfrac{\alpha-\psi}{2}+\tfrac{\beta-\psi}{2}+\tfrac{\gamma-\psi}{2}\right)$$

$$= \frac{\tan\dfrac{\alpha-\psi}{2}+\tan\dfrac{\beta-\psi}{2}+\tan\dfrac{\gamma-\psi}{2}-\tan\dfrac{\alpha-\psi}{2}\tan\dfrac{\beta-\psi}{2}\tan\dfrac{\gamma-\psi}{2}}{1-(\tan\dfrac{\alpha-\psi}{2}\tan\dfrac{\beta-\psi}{2}+\tan\dfrac{\beta-\psi}{2}\tan\dfrac{\gamma-\psi}{2}+\tan\dfrac{\gamma-\psi}{2}\tan\dfrac{\alpha-\psi}{2})}$$

$$= \frac{0-\dfrac{2\rho\sin(\phi-\psi)}{l}}{1-\dfrac{l+2\rho\cos(\phi-\psi)}{l}} = \tan(\phi-\psi),$$

which implies that

$$\frac{\alpha-\psi}{2}+\frac{\beta-\psi}{2}+\frac{\gamma-\psi}{2} = \phi-\psi \quad \text{or} \quad 2\phi = \alpha+\beta+\gamma-\psi. \quad \square$$

Example 4. Normals at α, β, γ, δ on the conic $l/r = 1+e\cos\theta$ meet on (ρ,ϕ). Then prove that

(a) $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}\tan\dfrac{\delta}{2}+\dfrac{(1+e)^2}{(1-e)^2}=0\,,$

(b) $\alpha+\beta+\gamma+\delta-2\phi=(2n+1)\pi.$

Solution. A normal to the conic is

$$\left(\frac{e\sin\lambda}{1+e\cos\lambda}\right)\frac{l}{r}=e\sin\theta+\sin(\theta-\lambda)=(e+\cos\lambda)\sin\theta-\sin\lambda\cos\theta$$

or

$$r\sin\theta(e+\cos\lambda)(1+e\cos\lambda)-r\cos\theta\sin\lambda(1+e\cos\lambda)=el\sin\lambda.$$

If the above normal passes through a point (ρ,ϕ), then

$$\rho\sin\phi(e+\cos\lambda)(1+e\cos\lambda)-\rho\cos\phi\sin\lambda(1+e\cos\lambda)=el\sin\lambda. \tag{1}$$

Using

$$\sin\lambda=\frac{2\tan\frac{\lambda}{2}}{1+\tan^2\frac{\lambda}{2}}=\frac{2t}{1+t^2}\quad\text{and}\quad\cos\lambda=\frac{1-\tan^2\frac{\lambda}{2}}{1+\tan^2\frac{\lambda}{2}}=\frac{1-t^2}{1+t^2},$$

where $\tan\frac{\lambda}{2}=t$, in the equation (1) we get

$$\rho\sin\phi\frac{(1+e)^2-(1-e)^2t^4}{(1+t^2)^2}-2\rho\cos\phi\frac{(1+e)t+(1-e)t^3}{(1+t^2)^2}=\frac{2elt}{1+t^2}$$

or

$$t^4((1-e)^2\rho\sin\phi)+2t^3(el+(1-e)\rho\cos\phi)+$$
$$+2t(el+(1+e)\rho\cos\phi)-(e^2+1)\rho\sin\phi=0.$$

Thus the roots of this equation are

$$t_1=\tan\frac{\alpha}{2},\quad t_2=\tan\frac{\beta}{2},\quad t_3=\tan\frac{\gamma}{2}\quad\text{and}\quad t_4=\tan\frac{\delta}{2}\,;$$

and we get

$$\begin{aligned}
s_1 &\equiv \sum t_1=-\frac{2(el+(1-e)\rho\cos\phi)}{(1-e)^2\rho\sin\phi}\,,\\
s_2 &\equiv \sum t_1t_2=0\,,\\
s_3 &\equiv \sum t_1t_2t_3=\frac{2(el+(1+e)\rho\cos\phi)}{(1-e)^2\rho\sin\phi}\,,\\
s_1 &\equiv t_1t_2t_3t_4=-\frac{(1+e)^2}{(1-e)^2}\,.
\end{aligned}$$

The last equation implies the first part. For the second part, we have

$$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=\frac{s_1-s_3}{1-s_2+s_4}=\cot\phi=\tan\left(\frac{\pi}{2}+\phi\right).$$

From the above equation we get

$$\frac{\alpha+\beta+\gamma+\delta}{2}=n\pi+\left(\frac{\pi}{2}+\phi\right),$$

which implies the second part. □

Example 5. Three normals are drawn from a point to a parabola $l/r=1+\cos\theta$. Show that the distance of the point from the focus of the parabola is equal to the diameter of the circumcircle of the triangle formed by tangents at the three feet of the normals.

Solution. The parabola is

$$\frac{l}{r}=1+\cos\theta. \tag{1}$$

A normal at λ to (1) is

$$\left(\frac{\sin\lambda}{1+\cos\lambda}\right)\frac{l}{r}=\sin(\theta-\lambda)+\sin\theta. \tag{2}$$

If (2) passes through a point $M(\rho,\phi)$, then

$$\left(\frac{l\sin\lambda}{1+\cos\lambda}\right)=\rho\sin(\phi-\lambda)+\rho\sin\phi=\rho(1+\cos\lambda)\sin\phi-\rho\sin\lambda\cos\phi.$$

or

$$\left(\frac{l\left(2\sin\frac{\lambda}{2}\cos\frac{\lambda}{2}\right)}{2\cos^2\frac{\lambda}{2}}\right)=\rho\left(2\cos^2\frac{\lambda}{2}\right)\sin\phi-2\rho\sin\frac{\lambda}{2}\cos\frac{\lambda}{2}\cos\phi$$

or

$$l\sin\frac{\lambda}{2}=\rho\left(2\cos^3\frac{\lambda}{2}\right)\sin\phi-2\rho\sin\frac{\lambda}{2}\cos^2\frac{\lambda}{2}\cos\phi$$

or

$$\sin\frac{\lambda}{2}\left(l+2\rho\cos\phi\cos^2\frac{\lambda}{2}\right)=2\rho\sin\phi\cos^3\frac{\lambda}{2}$$

or

$$\left(1-\cos^2\frac{\lambda}{2}\right)\left(l+2\rho\cos\phi\cos^2\frac{\lambda}{2}\right)^2=4\rho^2\sin^2\phi\cos^6\frac{\lambda}{2}$$

or

$$\left(1-\cos^2\frac{\lambda}{2}\right)\left(l^2+4l\rho\cos\phi\cos^2\frac{\lambda}{2}+4\rho^2\cos^2\phi\cos^4\frac{\lambda}{2}\right)$$
$$=4\rho^2\sin^2\phi\cos^6\frac{\lambda}{2}$$

or

$$4\rho^2\cos^6\frac{\lambda}{2}+(4l\rho\cos\phi+4\rho^2\cos^2\phi)\cos^4\frac{\lambda}{2}+$$
$$+\ \left(l^2-4l\rho\cos\phi\right)\cos^2\frac{\lambda}{2}-l^2=0. \qquad (3)$$

Thus the roots of this equation are $\cos^2(\alpha/2)$, $\cos^2(\beta/2)$ and $\cos^2(\gamma/2)$, where α, β, γ are the vectorial angles of the three feet P, Q, R of normals drawn from the point $M(\rho,\phi)$ to the parabola (1). From (3) we get

$$\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}=\frac{l^2}{4\rho^2}. \qquad (4)$$

Equations of tangents at the points $P\,(\alpha)$, $Q\,(\beta)$, $R\,(\gamma)$ are respectively

$$\frac{l}{r}=\cos\theta+\cos(\theta-\alpha)=2\cos\frac{\alpha}{2}\cos\left(\theta-\frac{\alpha}{2}\right), \qquad (5)$$

$$\frac{l}{r}=\cos\theta+\cos(\theta-2\beta)=2\cos\frac{\beta}{2}\cos\left(\theta-\frac{\beta}{2}\right), \qquad (6)$$

$$\frac{l}{r}=\cos\theta+\cos(\theta-2\gamma)=2\cos\frac{\gamma}{2}\cos\left(\theta-\frac{\gamma}{2}\right). \qquad (7)$$

The intersection point of (5) and (6) is $\left(\frac{l}{2}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}, \frac{\alpha+\beta}{2}\right)$.
The intersection point of (6) and (7) is $\left(\frac{l}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}, \frac{\beta+\gamma}{2}\right)$.
The intersection point of (7) and (5) is $\left(\frac{l}{2}\sec\frac{\gamma}{2}\sec\frac{\alpha}{2}, \frac{\gamma+\alpha}{2}\right)$.
Then, circle passing through these three points becomes

$$r = \left(\frac{l}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}\right)\cos\left(\theta - \frac{\alpha}{2} - \frac{\beta}{2} - \frac{\gamma}{2}\right). \tag{8}$$

The diameter of this circle is

$$\frac{l}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}. \tag{9}$$

From (4) and (9) the result follows. □

4.18 Problem Set 7

1. If the normal at P on a conic meets the major axis in G, then prove that $SG = e \cdot SP$.

2. If the normals at three points α, β, γ on the parabola $l/r = 1+\cos\theta$ meet in a point (ρ, ϕ), then show that $2\phi = \alpha+\beta+\gamma$.

3. If the normals at three points of the parabola $r = a\csc^2(\theta/2)$ whose vectorial angles are α, β, γ, meet in a point whose vectorial angle is ϕ, then prove that $2\phi = \alpha+\beta+\gamma-\pi$.

4. Prove that the portion of the normal to the conic $l/r = 1 + e\cos\theta$ at the point α, intercepted by the curve subtends at the pole an angle

$$2\arctan\left(\frac{e^2+2e\cos\alpha+1}{e\sin\alpha}\right).$$

5. If the normals to the parabola $l/r = 1 + \cos\theta$ at three points P, Q, R meet in a point M and S is the focus, then show that

$$2SP \cdot SQ \cdot SR = lSM^2.$$

6. Prove that the length of the shortest normal chord of the parabola $l/r = 1 + \cos\theta$ is $3l\sqrt{3}$ and that its inclination to the axis is $\arctan\sqrt{2}$.